Sobolev and Hardy-Sobolev spaces on graphs

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Abstract

Let Γ be a graph. Under suitable geometric assumptions on Γ, we give several equivalent characterizations of Sobolev and Hardy-Sobolev spaces on Γ, in terms of maximal functionals, Hajłasz type functionals or atomic decompositions. As an application, we study the boundedness of Riesz transforms on Hardy spaces on Γ. This gives the discrete counterpart of the corresponding results on Riemannian manifolds.

Contents

1 Introduction

1.1 The Euclidean case

Let $n \in \mathbb{N}^*$ and $1 \leq p \leq +\infty$. Throughout the paper, if $A(f)$ and $B(f)$ are two quantities depending on a function f ranging in a set E, say that $A(f) \lesssim B(f)$ if and only if there exists $C > 0$ such that, for all $f \in E$,

$$A(f) \leq CB(f),$$

and that $A(f) \sim B(f)$ if and only if $A(f) \lesssim B(f)$ and $B(f) \lesssim A(f)$.
The classical $W^{1,p}(\mathbb{R}^n)$ space, or its homogenous version $\dot{W}^{1,p}(\mathbb{R}^n)$, can be characterized in terms of maximal functions. Namely, if $f \in L^1_{loc}(\mathbb{R}^n)$, define, for all $x \in \mathbb{R}^n$,

$$Nf(x) := \sup_{B \ni x} \frac{1}{r(B)\,|B|} \int_B |f(y) - f_B|\, dy,$$

where the supremum is taken over all balls B containing x and

$$f_B := \frac{1}{|B|} \int_B f(y)dy$$

is the mean value of f over B. Here and after in this section, if $B \subset \mathbb{R}^n$ is a ball, $|B|$ stands for the Lebesgue measure of B and $r(B)$ for its radius.
Then ([Cal72]), for $1 < p \leq +\infty$, $\nabla f \in L^p(\mathbb{R}^n)$ if and only if $Nf \in L^p(\mathbb{R}^n)$, and

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \sim \|Nf\|_{L^p(\mathbb{R}^n)}.$$

Another maximal function characterizing Sobolev spaces was introduced in [ART05]. For $f \in L^1_{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$Mf(x) := \sup \left| \int_{\mathbb{R}^n} f(y) \text{div } \Phi(y) dy \right|,$$

where the supremum is taken over all vector fields $\Phi \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$, whose distributional divergence is a bounded function in $\mathbb{R}^n$, supported in a ball $B \subset \mathbb{R}^n$ containing x, with

$$\|\Phi\|_\infty + r(B)\,\|\text{div } \Phi\|_\infty \leq \frac{1}{|B|}.$$

Then ([ART05]), for $1 < p \leq +\infty$, $\nabla f \in L^p(\mathbb{R}^n)$ if and only if $Nf \in L^p(\mathbb{R}^n)$, and

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \sim \|Nf\|_{L^p(\mathbb{R}^n)}.$$

Another description of Sobolev spaces is due to Hajłasz. For $f \in L^1_{loc}(\mathbb{R}^n)$, $1 \leq p \leq +\infty$, say that $f \in \dot{M}^{1,p}(\mathbb{R}^n)$ if and only if there exists $g \in L^p(\mathbb{R}^n)$ such that, for all $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \leq d(x,y)(g(x) + g(y)). \tag{1.1}$$

Set

$$\|f\|_{\dot{M}^{1,p}(\mathbb{R}^n)} := \inf \|g\|_{L^p(\mathbb{R}^n)},$$

the infimum being taken over all functions g such that (1.1) holds. It was proved by Hajłasz ([Haj96]) that, for $1 < p \leq +\infty$, $f \in \dot{M}^{1,p}(\mathbb{R}^n)$ if and only if $\nabla f \in L^p(\mathbb{R}^n)$ and

$$\|f\|_{\dot{M}^{1,p}(\mathbb{R}^n)} \sim \|\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

What happens in these results when $p = 1$? The previous results break down when $p = 1$, but correct substitutes involving Hardy-Sobolev spaces can be given. More precisely (see below in the introduction), $\dot{M}^{1,1}(\mathbb{R}^n)$ coincides with the space of locally integrable functions with gradient in the $H^1(\mathbb{R}^n)$ Hardy space.
The $H^1(\mathbb{R}^n)$ Hardy space is well-known to be the right substitute for $L^1(\mathbb{R}^n)$ for many questions in harmonic analysis. Let us recall one possible definition of $H^1(\mathbb{R}^n)$. Fix a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. For all $t > 0$, define $\varphi_t(x) := t^{-n}\varphi\left(\frac{x}{t}\right)$. Define then $H^1(\mathbb{R}^n)$ as the space of locally integrable functions f on $\mathbb{R}^n$ such that the vertical maximal function

$$\mathcal{M}f(x) := \sup_{t>0} |\varphi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. Define

$$\|f\|_{H^1(\mathbb{R}^n)} := \|\mathcal{M}f\|_{L^1(\mathbb{R}^n)}.$$

As for classical Sobolev spaces, let us consider the Hardy-Sobolev space $H^{1,1}(\mathbb{R}^n)$ made of functions $f \in L^1(\mathbb{R}^n)$ such that $\nabla f \in H^1(\mathbb{R}^n)$, in the sense that, for all $1 \leq j \leq n$, $\frac{\partial f}{\partial x_j} \in H^1(\mathbb{R}^n)$. Define also $\dot{H}^{1,1}(\mathbb{R}^n)$ as the space of functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\nabla f \in H^1(\mathbb{R}^n)$, equipped with the semi-norm

$$\|f\|_{\dot{H}^{1,1}(\mathbb{R}^n)} := \|\nabla f\|_{H^1(\mathbb{R}^n)}.$$

Various characterizations of this space (as well as its adaptations to the case of domains of $\mathbb{R}^n$) were given in the literature. It can be described in terms of a functional involving second order differences ([Str90]). In [Miy90], $H^{1,1}(\mathbb{R}^n)$ was characterized in terms of the maximal function Nf. Namely, for $f \in L^1_{loc}(\mathbb{R}^n)$, $\nabla f \in H^1(\mathbb{R}^n)$ if and only if $Nf \in L^1(\mathbb{R}^n)$ and

$$\|Nf\|_{L^1(\mathbb{R}^n)} \sim \|\nabla f\|_{H^1(\mathbb{R}^n)} := \sum_{j=1}^{n} \left\| \frac{\partial f}{\partial x_j} \right\|_{H^1(\mathbb{R}^n)}.$$

It was shown in [ART05] that the functional Mf defined above characterizes Hardy-Sobolev spaces (actually, this was the reason why this maximal function was introduced in [ART05], since it is particularly suited to the study of Hardy-Sobolev spaces on strongly Lipschitz domains of $\mathbb{R}^n$). More precisely, $\nabla f \in H^1(\mathbb{R}^n)$ if and only if $Mf \in L^1(\mathbb{R}^n)$ and

$$\|Mf\|_{L^1(\mathbb{R}^n)} \sim \|\nabla f\|_{H^1(\mathbb{R}^n)}.$$

Moreover, going back to Hajłasz's functional, it was proved in [KS08] that $f \in \dot{M}^{1,1}(\mathbb{R}^n)$ if and only if $\nabla f \in H^1(\mathbb{R}^n)$ and

$$\|f\|_{\dot{M}^{1,1}(\mathbb{R}^n)} \sim \|\nabla f\|_{H^1(\mathbb{R}^n)}.$$

Finally, an atomic decomposition for Hardy-Sobolev spaces was given in [Str90]. In this paper, an atom is a function b supported in a cube such that $(-\Delta)^{1/2}b$ satisfies suitable L^p estimates ([Str90], definition 5.1).
Another characterization of $H^1(\mathbb{R}^n)$ states that it is exactly the space of functions $f \in L^1(\mathbb{R}^n)$ such that, for all $1 \le j \le n$, $\frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f \in L^1(\mathbb{R}^n)$ (see [FS72]). The operators $R_j := \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} f$ are the Riesz transforms. Thus, $(-\Delta)^{-1/2}$ maps continuously $H^1(\mathbb{R}^n)$ into $\dot{H}^{1,1}(\mathbb{R}^n)$.

1.2 The case of Riemannian manifolds

These various characterizations can be extended to the framework of Riemannian manifolds. Namely, let M be a complete Riemannian manifold, endowed with its Riemannian metric d and its Riemannian measure μ. Say that M satisfies the doubling condition if there exists $C > 0$ such that, for all $x \in M$ and all $r > 0$,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)).$$

Say that M satisfies an L^1 scaled Poincaré inequality on balls if there exists $C > 0$ such that, for all balls $B \subset M$ with radius r and all functions $f \in C^\infty(B)$,

$$\int_B |f(x) - f_B| \, d\mu(x) \le Cr \int_B |df(x)| \, d\mu(x).$$

Define the $\dot{M}^{1,p}$ spaces and the Nf functional as in the Euclidean case. Then, for $1 \le p < +\infty$, $f \in \dot{M}^{1,p}$ if and only if $Nf \in L^p(M)$ ([KT07]). A version of the maximal function in [ART05] is given in [BD11], where it is shown that it characterizes $\dot{M}^{1,1}$. Moreover, an atomic decomposition for $\dot{M}^{1,1}$ is provided in [BD10], where it is also shown that $f \in \dot{M}^{1,1}$ if and only if

df belongs to the Hardy space of exact differential forms $H^1_d(\Lambda^1 T^*M)$ introduced in [AMR08]. Since $d\Delta^{-1/2}$ is bounded from $H^1_{d^*}(\Lambda^0 T^*M)$ from $H^1_d(\Lambda^1 T^*M)$ (see [AMR08], Theorem 5.16), if Δ denotes the Laplace-Beltrami operator, $\Delta^{-1/2}$ maps continuously $H^1_{d^*}(\Lambda^0 T^*M)$ into $\dot{M}^{1,1}$.

In the present work, we investigate Sobolev and Hardy-Sobolev spaces on graphs, and establish the discrete counterpart of the results obtained on Riemannian manifolds. Namely, we characterize Sobolev and Hardy-Sobolev spaces in terms of maximal functions and provide an atomic decomposition for Hardy-Sobolev spaces. We also investigate the boundedness of Riesz transforms on Hardy spaces.

2 Description of the results

2.1 Presentation of the graph

The geometric context is the same as in [BR09], and we recall it for the sake of completeness. Let Γ be an infinite set and $\mu_{xy} = \mu_{yx}$ a symmetric weight on $\Gamma \times \Gamma$. Say that $x \sim y$ if and only if $\mu_{xy} > 0$, and let E stand for the set of edges in Γ, defined as the set of $(x, y) \in \Gamma \times \Gamma$ such that $\mu_{xy} > 0$. For all $x \in \Gamma$, say that x is a vertex of Γ.
For $x, y \in \Gamma$, a path joining x to y is a finite sequence of vertices $x_0 = x, \cdots, x_N = y$ such that, for all $0 \leq i \leq N-1, x_i \sim x_{i+1}$. Say that this path has length N. Assume that Γ is connected, which means that, for all $x, y \in \Gamma$, there exists a path joining x to y. The distance between x and y, denoted $d(x, y)$, is defined as the shortest length of a path joining x and y. For all $x \in \Gamma$ and all $r \geq 0$, define the closed ball

$$B(x, r) := \{y \in \Gamma; d(x, y) \leq r\}.$$

In the sequel, we always assume that Γ is locally uniformly finite, which means that there exists $N \in \mathbb{N}^*$ such that, for all $x \in \Gamma, \#B(x, r) \leq N$.
For any subset $\Omega \subset \Gamma$, set

$$\partial\Omega := \{x \in \Omega; \exists y \sim x, y \notin \Omega\}$$

and

$$\overset{\circ}{\Omega} := \Omega \setminus \partial\Omega.$$

In other words, $\overset{\circ}{\Omega}$ is the set of points $x \in \Omega$ such that $y \in \Omega$ whenever $x \sim y$. Denote by E_Ω the set of edges in Ω,

$$E_\Omega = \{(x, y) \in \Omega \times \Omega : x \sim y, x, y \in \Omega\}.$$

We also define a distance on E. For $\gamma = (x, y)$ and $\gamma' = (x', y') \in E$, set

$$d(\gamma, \gamma') := \max\left(d(x, x'), d(y, y')\right).$$

2.1.1 The measures on Γ and E

For all $x \in \Gamma$, set $m(x) = \sum\limits_{y \sim x} \mu_{xy}$ (recall that this sum has at most N terms). We always assume in the sequel that $m(x) > 0$ for all $x \in \Gamma$. If $\Omega \subset \Gamma$, define $m(\Omega) = \sum\limits_{x \in \Omega} m(x)$. For all

$x \in \Gamma$ and $r > 0$, write $V(x, r)$ instead of $m(B(x, r))$ and, if B is a ball, $m(B)$ will be denoted by $V(B)$.
Here is a growth assumption on the volume of balls of Γ, which may be satisfied or not.

Definition 2.1 *[Doubling property] Say that (Γ, d, m) satisfies the doubling property if there exists a constant $C > 0$ such that for all balls $B(x, r), x \in \Gamma, r > 0$,*

$$V(x, 2r) \leq CV(x, r). \tag{D}$$

This means that (Γ, d, m) is a space of homogeneous type in the sense of Coifman and Weiss ([CW77]). It is plain to check that, if Γ satisfies (D), then there exist $C, s > 0$ such that, for all $x \in \Gamma$, all $r > 0$ and all $\theta \geq 1$,

$$V(x, \theta r) \leq C\theta^s V(x, r). \tag{2.3}$$

Remark 2.2 *Observe also that, since Γ is infinite, it is also unbounded (since it is locally uniformly finite) so that, if (D) holds, then $m(\Gamma) = +\infty$ (see [Mar01]).*

For all $1 \leq p < +\infty$, say that a function $f : \Gamma \to \mathbb{R}$ belongs to $L^p(\Gamma)$ if

$$\|f\|_{L^p(\Gamma)} = \left(\sum_{x \in \Gamma} |f(x)|^p m(x)\right)^{1/p} < +\infty.$$

Note that the $L^2(\Gamma)$-norm derives from the scalar product

$$\langle f, g \rangle_{L^2(\Gamma)} := \sum_{x \in \Gamma} f(x) g(x) m(x).$$

Say that $f \in L^\infty(\Gamma)$ if

$$\|f\|_{L^\infty(\Gamma)} = \sup_{x \in \Gamma} |f(x)| < +\infty.$$

If $B \subset \Gamma$ is a ball, denote by $L^p_0(B)$ the subspace of $L^p(\Gamma)$ made of functions f supported in B and satisfying

$$\sum_{x \in B} f(x) m(x) = 0.$$

We also need a measure on E. For any subset $A \subset E$, define

$$\mu(A) := \sum_{(x,y) \in A} \mu_{xy}.$$

It is easily checked ([BR09], Section 8) that, if (D) holds, then E, equipped with the distance d and the measure μ, is a space of homogeneous type.
Define L^p spaces on E in the following way. For $1 \leq p < +\infty$, say that a function F on E belongs to $L^p(E)$ if and only if F is antisymmetric, which means that $F(x, y) = -F(y, x)$ for all $(x, y) \in E$, and

$$\|F\|^p_{L^p(E)} := \frac{1}{2} \sum_{(x,y) \in E} |F(x, y)|^p \mu_{xy} < +\infty.$$

Observe that the $L^2(E)-$norm derives from the scalar product

$$\langle F, G\rangle_{L^2(E)} := \frac{1}{2}\sum_{x,y\in\Gamma} F(x,y)G(x,y)\mu_{xy}.$$

Finally, say that $F \in L^\infty(E)$ if and only if F is antisymmetric and

$$\|F\|_{L^\infty(E)} := \frac{1}{2}\sup_{(x,y)\in E} |F(x,y)| < +\infty.$$

Define $L^p(E_\Omega)$ similarly.

2.1.2 The Markov kernel

Define $p(x,y) = \frac{\mu_{xy}}{m(x)}$ for all $x, y \in \Gamma$. Observe that $p(x,y) = 0$ if $d(x,y) \geq 2$. Moreover, for all $x \in \Gamma$,

$$\sum_{y\in\Gamma} p(x,y) = 1 \tag{2.4}$$

and for all $x, y \in \Gamma$,

$$p(x,y)m(x) = p(y,x)m(y). \tag{2.5}$$

Another assumption on (Γ, μ) which will be used in the sequel is a uniform lower bound for $p(x,y)$ when $x \sim y$. For $\alpha > 0$, say that (Γ, μ) satisfies the condition $\Delta(\alpha)$ if, for all $x, y \in \Gamma$,

$$(x \sim y \Leftrightarrow \mu_{xy} \geq \alpha m(x)) \text{ and } x \sim x. \tag{$\Delta(\alpha)$}$$

For all functions f on Γ and all $x \in \Gamma$, define

$$Pf(x) = \sum_{y\in\Gamma} p(x,y)f(y).$$

It is easily checked ([BR09]), using (2.5), that, for all functions f on Γ,

$$\langle (I-P)f, f\rangle = \frac{1}{2}\sum_{x,y} p(x,y)|f(x) - f(y)|^2 m(x). \tag{2.6}$$

Identity (2.6) leads to the definition of the operator "length of the gradient" by

$$\nabla f(x) = \left(\frac{1}{2}\sum_{y\in\Gamma} p(x,y)|f(y) - f(x)|^2\right)^{1/2},$$

so that, for all functions f on Γ,

$$\langle (I-P)f, f\rangle_{L^2(\Gamma)} = \|\nabla f\|^2_{L^2(\Gamma)}. \tag{2.7}$$

2.1.3 The differential and divergence operators

We now define a discrete differential, following the definitions of [BR09] but dealing with functions defined on subsets of Γ. Let $\Omega \subset \Gamma$. For any function $f : \Omega \to \mathbb{R}$ and any $\gamma = (x, y) \in E_\Omega$, define

$$df(\gamma) = f(y) - f(x). \tag{2.8}$$

The function df is clearly antisymmetric on E_Ω. Moreover, it is easily checked ([BR09], p.313) that, if $(\Delta(\alpha))$ holds, then for all $p \in [1, +\infty]$ and all functions f on Γ,

$$\|df\|_{L^p(E)} \sim \|\nabla f\|_{L^p(\Gamma)}. \tag{2.9}$$

We define now a divergence operator in such a way that a discrete integration by parts formula holds (see [BR09]). Let F be any (antisymmetric) function in $L^2(E_\Omega)$. If f is a function on Ω vanishing on $\partial\Omega$ such that $df \in L^2(E_\Omega)$, one has

$$\begin{aligned}
\langle df, F\rangle_{L^2(E_\Omega)} &= \frac{1}{2} \sum_{x,y\in\Omega,\ x\sim y} df(x,y)F(x,y)\mu_{xy} \\
&= -\sum_{x,y\in\Omega,\ x\sim y} f(x)F(x,y)\mu_{xy} \\
&= -\sum_{x\in\overset{\circ}{\Omega}} f(x)\left(\sum_{y\sim x,\ y\in\Gamma} p(x,y)F(x,y)\right) m(x),
\end{aligned}$$

where the second line is due to the fact that F is antisymmetric and the third one holds because $f(x) = 0$ when $x \in \partial\Omega$ and all the neighbours of x in Γ actually belong to Ω when $x \in \overset{\circ}{\Omega}$. Thus, if we define the divergence of F by

$$\delta F(x) := \sum_{y\sim x,\ y\in\Gamma} p(x,y)F(x,y)$$

for all $x \in \overset{\circ}{\Omega}$, it follows that

$$\langle df, F\rangle_{L^2(E_\Omega)} = -\langle f, \delta F\rangle_{L^2(\overset{\circ}{\Omega})}. \tag{2.10}$$

Remark 2.3 *A slightly different integration by parts formula on graphs can be found in [CGZ05], formula 2.4.*

2.1.4 The Poincaré inequality on balls

Definition 2.4 *[L^p Poincaré inequality on balls] Let $p \in [1, +\infty)$. Say that Γ satisfies an L^p scaled Poincaré inequality on balls if there exists a constant $C > 0$ such that, for all functions f on Γ and all balls $B \subset \Gamma$ of radius $r > 0$,*

$$\sum_{x\in B} |f(x) - f_B|^p m(x) \le Cr^p \sum_{x\in B} |\nabla f(x)|^p m(x), \tag{P_p}$$

where

$$f_B = \frac{1}{V(B)} \sum_{x\in B} f(x)m(x). \tag{2.11}$$

Remark 2.5 1. *Note that, if (P_1) holds, then one has an L^p Poincaré inequality for all $p \in [1, +\infty)$ (see [HK00]).*

2. *Moreover, if (P_p) holds for some $p \in (1, +\infty)$, there exists $q < p$ such that (P_q) still holds ([KZ08]).*

2.2 Sobolev spaces

Let Γ be a graph as in Section 2.1. Let $1 \leq p \leq +\infty$. Say that a scalar-valued function f on Γ belongs to the Sobolev space $W^{1,p}(\Gamma)$ if and only if

$$\|f\|_{W^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \|\nabla f\|_{L^p(\Gamma)} < +\infty.$$

As in [BR09] we will also consider the homogeneous versions of Sobolev spaces. Define $\dot{W}^{1,p}(\Gamma)$ as the space of all scalar-valued functions f on Γ such that $\nabla f \in L^p(\Gamma)$, equipped with the semi-norm

$$\|f\|_{\dot{W}^{1,p}(\Gamma)} := \|\nabla f\|_{L^p(\Gamma)}.$$

If B is any ball in Γ and $1 \leq p \leq +\infty$, denote by $W_0^{1,p}(B)$ the subspace of $W^{1,p}(\Gamma)$ made of functions supported in $\overset{\circ}{B}$.

2.3 Characterizations of Sobolev spaces

In the present section, we give various characterizations of Sobolev spaces on graphs. The first one is formulated in terms of Hajłasz's functionals (see [Haj03b, HK00]):

Definition 2.6 *Let $1 \leq p \leq +\infty$.*

1. *The inhomogeneous Sobolev space $M^{1,p}(\Gamma)$ is defined as the space of all functions $f \in L^p(\Gamma)$ such that there exists a non-negative function $g \in L^p(\Gamma)$ satisfying*

$$|f(x) - f(y)| \leq d(x,y)\,(g(x) + g(y)) \text{ for all } x, y \in \Gamma. \tag{2.12}$$

We equip $M^{1,p}(\Gamma)$ with the norm

$$||f||_{M^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \inf_g ||g||_{L^p(\Gamma)}, \tag{2.13}$$

where the infimum is taken over all functions $g \in L^p(\Gamma)$ such that (2.12) holds.

2. *The homogeneous Sobolev space $\dot{M}^{1,p}(\Gamma)$ is defined as the space of all functions f on Γ such that there exists a non-negative function $g \in L^p(\Gamma)$ satisfying (2.12). We equip $\dot{M}^{1,p}(\Gamma)$ with the semi-norm*

$$||f||_{\dot{M}^{1,p}(\Gamma)} = \inf_g ||g||_{L^p(\Gamma)},$$

where the infimum is taken over all functions $g \in L^p(\Gamma)$ such that (2.12) holds.

Remark 2.7 *If $B \subset \Gamma$ is a ball, define $M^{1,p}(B)$ and $\dot{M}^{1,p}(B)$, replacing Γ by B in Definition 2.6.*

We will also characterize Sobolev spaces in terms of two maximal functions.
The first maximal function is modelled on the one in [Cal72]. For all functions f on Γ and all $x \in \Gamma$, define $Nf(x)$ by

$$Nf(x) := \sup_{B \ni x} \frac{1}{r(B)V(B)} \sum_{y \in B} |f(y) - f_B| m(y) \tag{2.14}$$

where the supremum is taken over all balls B with radius $r(B) > 0$ and f_B denotes the mean value of f on B defined by (2.11).

Remark 2.8 *For further use, observe that, if f is a non-constant function on Γ, then $Nf(x) \neq 0$ for all $x \in \Gamma$. Indeed, if $Nf(x) = 0$ for some $x \in \Gamma$, then $f(y) = f_B$ for all balls B containing x. Thus, f is constant on any ball containing x, therefore constant on Γ.*

The second maximal function we use is inspired by [ART05] and [BD11]. Its definition involves estimates on the (discrete) divergence of test functions. More precisely, for all function f on Γ, define, for all $x \in \Gamma$,

$$\mathcal{M}^{+}(f)(x) = \sup_{F} \left| \sum_{y \in \overset{\circ}{B}} f(y)(\delta F)(y) m(y) \right|, \tag{2.15}$$

where the supremum is taken over all balls $B \subset \Gamma$ containing x and all antisymmetric functions $F : E \to \mathbb{R}$ supported in E_B and satisfying

$$\|F\|_{L^\infty(E_B)} \leq \frac{1}{V(B)}, \qquad \|\delta F\|_{L^\infty(\overset{\circ}{B})} \leq \frac{1}{r(B)V(B)}. \tag{2.16}$$

Define now, for $1 \leq p \leq +\infty$,

$$S^{1,p}(\Gamma) := \{f \in L^p(\Gamma); Nf \in L^p(\Gamma)\},$$

equipped with the norm

$$\|f\|_{S^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \|Nf\|_{L^p(\Gamma)}.$$

Consider also the $\dot{S}^{1,p}(\Gamma)$ space, made of functions f on Γ such that $Nf \in L^p(\Gamma)$, equipped with the semi-norm

$$\|f\|_{\dot{S}^{1,p}(\Gamma)} := \|Nf\|_{L^p(\Gamma)}.$$

Define also

$$E^{1,p}(\Gamma) := \left\{f \in L^p(\Gamma); \mathcal{M}^{+} f \in L^p(\Gamma)\right\},$$

equipped with the norm

$$\|f\|_{E^{1,p}(\Gamma)} := \|f\|_{L^p(\Gamma)} + \left\|\mathcal{M}^{+} f\right\|_{L^p(\Gamma)},$$

as well as its homogenous version.
Our first result is that, under (D), $(\Delta(\alpha))$ and (P_p), the spaces $W^{1,p}(\Gamma)$, $S^{1,p}(\Gamma)$, $E^{1,p}(\Gamma)$ and $M^{1,p}(\Gamma)$, as well as their homogenous versions, coincide:

Theorem 2.9 *Let $1 < p \leq +\infty$. Assume that Γ satisfies (D), $(\Delta(\alpha))$ and (P_p). Then:*

1. $W^{1,p}(\Gamma) = S^{1,p}(\Gamma) = E^{1,p}(\Gamma) = M^{1,p}(\Gamma)$,
2. $\dot{W}^{1,p}(\Gamma) = \dot{S}^{1,p}(\Gamma) = \dot{E}^{1,p}(\Gamma) = \dot{M}^{1,p}(\Gamma)$.

2.4 Characterization of Hardy-Sobolev spaces

When $p = 1$, as in the Euclidean case recalled in the introduction, the conclusion of Theorem 2.9 does not hold. The following example is inspired by [Haj03a], Example 3. Take $\Gamma = \mathbb{Z}$ with its usual metric. Define, for all $x \in \mathbb{Z}$,

$$f(x) := \begin{cases} \dfrac{x}{|x| \ln |x|} & \text{if } |x| \geq 2, \\ 0 & \text{if } |x| \leq 1. \end{cases}$$

Then $f \in \dot{W}^{1,1}(\mathbb{Z})$. Indeed, for all $x \geq 2$, the mean-value theorem yields

$$|f(x+1) - f(x)| = \left| \frac{1}{\ln x} - \frac{1}{\ln(x+1)} \right| \leq \frac{1}{x\,(\ln x)^2}.$$

As a consequence, for all $x \geq 3$,

$$|\nabla f(x)| \leq \frac{C}{|x|\,(\ln |x|)^2}. \tag{2.17}$$

Since f is odd, (2.17) also holds for all $x \leq -3$. As a consequence,

$$\sum_{x \in \mathbb{Z}} |\nabla f(x)| < +\infty.$$

Assume now that there exists a non-negative function $g \in L^1(\mathbb{Z})$ such that $|f(x) - f(y)| \leq d(x,y)\,(g(x) + g(y))$ for all $x, y \in \mathbb{Z}$. Then, for all $x \geq 3$,

$$|f(x) - f(-x)| \leq 2x\,(g(x) + g(-x)).$$

Since f is odd, this means that, for all $x \geq 3$,

$$\frac{1}{x}\,|f(x)| \leq (g(x) + g(-x)).$$

Therefore,

$$2 \sum_{|x| \geq 3} g(x) \geq \sum_{x \geq 3} \frac{1}{x \ln x} = +\infty,$$

which contradicts the fact that $g \in L^1(\mathbb{Z})$.

The goal of this section is to give an endpoint version of Theorem 2.9 when $p = 1$. We will focus on the case of homogenous spaces. As it will turn out, asssuming (D) and (P_1), one still has $\dot{M}^{1,1}(\Gamma) = \dot{S}^{1,1}(\Gamma)$. Two extra characterizations of $\dot{M}^{1,1}(\Gamma)$ will be given: the first one is formulated in terms of $\mathcal{M}^+ f$, the second one is an atomic decomposition. We first introduce these new descriptions.

2.4.1 Maximal Hardy-Sobolev space

It turns out that, as in the Euclidean case and in the context of Riemannian manifolds (see the introduction), Hardy-Sobolev spaces on Γ can be defined by means of the functional $\mathcal{M}^+$. Let us first give a definition:

Definition 2.10 ***(Maximal Hardy-Sobolev space)***

1. *We define the Hardy-Sobolev space $HS^1_{\max}(\Gamma)$ as follows:*

$$HS^1_{\max}(\Gamma) = \{f \in L^1(\Gamma) : \mathcal{M}^+ f \in L^1(\Gamma)\}. \tag{2.18}$$

   This space is equipped with the norm

$$||f||_{HS^1_{\max}(\Gamma)} := \|f\|_{L^1(\Gamma)} + \left\|\mathcal{M}^+ f\right\|_{L^1(\Gamma)}. \tag{2.19}$$

2. *The homogenous Hardy-Sobolev space $\dot{H}S^1_{\max}(\Gamma)$ is the space of all functions f on Γ such that $\mathcal{M}^+ f \in L^1(\Gamma)$. It is equipped with the semi-norm*

$$||f||_{\dot{H}S^1_{\max}(\Gamma)} := \left\|\mathcal{M}^+ f\right\|_{L^1(\Gamma)}.$$

2.4.2 Atomic Hardy-Sobolev spaces

Definition 2.11 *For $1 < t \leq +\infty$, define t' by $\frac{1}{t} + \frac{1}{t'} = 1$. Say that a function a on Γ is a homogeneous Hardy-Sobolev $(1,t)$ – atom if*

1. *a is supported in a ball B,*
2. $||\nabla a||_t \leq V(B)^{-\frac{1}{t'}}$,
3. $\sum_{x\in\Gamma} a(x)m(x) = 0$.

If f is a function on Γ, say that $f \in \dot{H}S^1_{t,ato}(\Gamma)$ if there exist a sequence $(\lambda_i)_{i\geq 1} \in l^1$ and a sequence of homogeneous Hardy-Sobolev $(1,t)$-atoms such that

$$f = \sum_i \lambda_i a_i. \tag{2.20}$$

This space is equipped with the semi-norm

$$||f||_{\dot{H}S^1_{t,ato}(\Gamma)} = \inf \sum_i |\lambda_i|$$

where the infimum is taken over all possible decompositions.
Notice that the convergence in (2.20) is required to hold in $\dot{W}^{1,1}(\Gamma)$, which means that

$$\lim_{k\to+\infty} \left\|\nabla \left(f - \sum_{j=0}^{k} \lambda_j a_j\right)\right\|_{L^1(\Gamma)} = 0.$$

The link between convergence in (2.20) and pointwise convergence will be made explicit in Proposition 5.2 below.
In the sequel, we will establish:

Theorem 2.12 *Assume that* (D), $(\Delta(\alpha))$ *and* (P_1) *hold. Then* $\dot{S}^{1,1}(\Gamma) = \dot{M}^{1,1}(\Gamma) = \dot{H}S^1_{\max}(\Gamma) = \dot{H}S^1_{t,ato}(\Gamma)$ *for all* $t \in (1, +\infty]$. *In particular,* $\dot{H}S^1_{t,ato}(\Gamma)$ *does not depend on* t.

Remark 2.13 *Assume that, in Definition 2.11, we replace condition* 3 *by*

$3'$ $\|a\|_{L^t(B)} \leq rV(B)^{-\frac{1}{t'}}$,

where r *is the radius of* B, *and we define* $\dot{H}S^1_{t,ato}(\Gamma)$ *as before, using this new type of atoms. Then, as the proof of Theorem 2.12 will show (see Remark 5.5 below), we obtain exactly the same* $\dot{H}S^1_{t,ato}(\Gamma)$ *space. This remark (inspired by ideas in [BD10]) will turn out to be important for the study of Riesz transforms.*

2.5 Interpolation

As a consequence of the characterization of Hardy-Sobolev and Sobolev spaces through maximal functions, we establish an interpolation result between Hardy-Sobolev and Sobolev spaces:

Theorem 2.14 *Let* $1 < q \leq +\infty$ *and* $\theta \in (0,1)$. *Define* p *such that* $\frac{1}{p} = (1-\theta) + \frac{\theta}{q}$. *Then, for the complex interpolation method,*

$$\left[\dot{S}^{1,1}(\Gamma), \dot{W}^{1,q}(\Gamma)\right]_\theta = \dot{W}^{1,p}(\Gamma).$$

2.6 Riesz transforms

The Riesz transform in our context is the operator $R := d(I-P)^{-1/2}$, which maps functions on Γ to functions on E. The equality (2.7) shows that R is $L^2(\Gamma) - L^2(E)$ bounded. For $1 < p < +\infty$, the L^p-boundedness of R was investigated in [BR09] under various assumptions[1]. In particular, under (D) and the Poincaré inequality (P_2), R is $L^p(\Gamma) - L^p(E)$ bounded for all $1 < p \leq 2$ (and even under weaker assumptions, see [Rus00]).
For $p = 1$, the Riesz transform is not $L^1(\Gamma) - L^1(E)$ bounded, but an endpoint version of the L^p-boundedness of R for $1 < p \leq 2$ was proved in [Rus01]. This endpoint version involves the $H^1(\Gamma)$ atomic Hardy space on Γ, the definition of which we recall now. An atom in $H^1(\Gamma)$ is a function $a \in L^2(\Gamma)$, supported in a ball $B \subset \Gamma$ and satisfying

$$\sum_{x \in \Gamma} a(x)m(x) = 0 \text{ and } \|a\|_{L^2(\Gamma)} \leq V(B)^{-1/2}.$$

A function f on Γ is said to belong to $H^1(\Gamma)$ if and only if there exist a sequence $(\lambda_j)_{j\geq 1} \in l^1$ and a sequence of atoms $(a_j)_{j \geq 1}$ such that

$$f = \sum_j \lambda_j a_j,$$

[1]Observe that the L^p-boundedness results of [BR09] are stated for the operator $\nabla(I-P)^{-1/2}$, but (2.9) shows at once that analogous conclusions hold for $d(I-P)^{-1/2}$.

where the series converges in $L^1(\Gamma)$. In this case, define

$$\|f\|_{H^1(\Gamma)} := \inf \sum_j |\lambda_j|,$$

where, as usual, the infimum is taken over all possible decompositions of f.
Under (D) and (P_2), the Riesz transform is $H^1(\Gamma) - L^1(E)$ bounded ([Rus01]). This means that $(I-P)^{-1/2}$ is bounded from $H^1(\Gamma)$ to $\dot{W}^{1,1}(\Gamma)$. Here, under an extra assumption on the volume growth of balls of Γ, we prove that $(I-P)^{-1/2}$ maps continuously $H^1(\Gamma)$ into $\dot{S}^{1,1}(\Gamma)$:

Theorem 2.15 *Assume that Γ satisfies (D) and (P_2). Assume furthermore that there exist $C>0$ and $d\geq 1$ such that, for all $x\in\Gamma$ and all $1\leq r\leq s$,*

$$\frac{V(x,r)}{V(x,s)} \leq C\left(\frac{r}{s}\right)^d. \tag{2.21}$$

Then $(I-P)^{-1/2}$ is bounded from $H^1(\Gamma)$ into $\dot{S}^{1,1}(\Gamma)$.

Remark 2.16 *Under (D), there exists $C'>0$ such that, for all $x\in\Gamma$ and all $r\geq 1$,*

$$V(x,C'r) \geq 2V(x,r)$$

(see [CG98], Lemma 2.2). This implies that (2.21) always holds with some $d>0$. In Theorem 2.15, we assume furthermore that $d\geq 1$. This technical assumption seems to be required by our argument (see the proof of Theorem 2.15 in Section 6 below), and could probably be removed. Note that assumption (2.21) is satisfied when, for instance, $V(x,r)\sim r^d$ for some $d\geq 1$, which holds when Γ is the Cayley graph of a group with polynomial volume growth.

3 Proofs of the characterizations of Sobolev spaces

This section is devoted to the proof of Theorem 2.9. It will be convenient to use the following observation:

Lemma 3.1 *For all functions f on Γ, all $x\in\Gamma$ and all $r\geq 0$,*

$$\left|f(x) - f_{B(x,r)}\right| \leq CrNf(x). \tag{3.22}$$

Proof of Lemma 3.1: first, the conclusion is trivial when $0\leq r<1$, since in this case, $B(x,r)=\{x\}$ so that the left-hand side of (3.22) vanishes. Assume now that $r\geq 1$ and let $j\in\mathbb{N}$ be the integer such that $2^j\leq r<2^{j+1}$. Define $B:=B(x,2^{j+1})$ and, for all $-1\leq i\leq j+1$, $B_i=B(x,2^i)$, so that $B=B_{j+1}$. Since $f(x)=f_{B\left(x,\frac{1}{2}\right)}$,

$$\begin{aligned}
|f(x)-f_B| &\leq \sum_{i=-1}^{j} \left|f_{B_i}-f_{B_{i+1}}\right| \\
&\leq \sum_{i=-1}^{j} \frac{1}{V(B_i)} \sum_{y\in B_i} |f(y)-f_{B_{i+1}}|m(y) \\
&\leq C\sum_{i=-1}^{j} \frac{r(B_{i+1})}{r(B_{i+1})V(B_{i+1})} \sum_{y\in B_{i+1}} |f(y)-f_{B_{i+1}}|m(y) \\
&\leq C2^jNf(x),
\end{aligned} \tag{3.23}$$

where the third line uses (D). Moreover, since $B(x,r) \subset B$,

$$\begin{aligned} \left| f_{B(x,r)} - f_B \right| &\leq \frac{1}{V(x,r)} \sum_{y \in B(x,r)} |f(y) - f_B| \, m(y) \\ &\leq C \frac{1}{V(B)} \sum_{y \in B} |f(y) - f_B| \, m(y) \\ &\leq C 2^j N f(x), \end{aligned} \tag{3.24}$$

and the conjunction of (3.23) and (3.24) yields the conclusion (note that we used (D) again in the second line). □

As a corollary, one has (see also Lemma 3.6 in [HK98]):

Proposition 3.2 *For all functions f on Γ and all $x, y \in \Gamma$,*

$$|f(x) - f(y)| \lesssim d(x,y) \left(Nf(x) + Nf(y) \right).$$

Proof: let $x, y \in \Gamma$ with $x \neq y$ and $r := d(x,y)$. Lemma 3.1 yields

$$\left| f(x) - f_{B(x,r)} \right| \leq CrNf(x). \tag{3.25}$$

On the other hand, since $B(x,r) \subset B(y,2r)$, using Lemma 3.1 again, one obtains

$$\begin{aligned} |f(y) - f_{B(x,r)}| &\leq |f(y) - f_{B(y,2r)}| + |f_{B(y,2r)} - f_{B(x,r)}| \\ &\leq CrNf(y) + \frac{1}{V(x,r)} \sum_{z \in B(x,r)} \left| f(z) - f_{B(y,2r)} \right| m(z) \\ &\leq CrNf(y) + C \frac{1}{V(y,2r)} \sum_{z \in B(y,2r)} \left| f(z) - f_{B(y,2r)} \right| m(z) \\ &\leq CrNf(y). \end{aligned} \tag{3.26}$$

Thus, (3.25) and (3.26) yield the desired result. □

To establish that Sobolev spaces can also be characterized in terms of $\mathcal{M}^+ f$, we have to solve the equation $\delta F = g$ in L^∞ spaces (see also [BD11], Proposition 5.1 and [DMRT10] for the original ideas):

Proposition 3.3 *Assume that Γ satisfies (D) and (P_1). Let B a ball of Γ with $r(B) \geq 1$ and $g \in L^\infty_0(B)$. Then, there exists $F \in L^\infty(E_B)$ such that $\delta F = g$ in $\overset{\circ}{B}$ and*

$$||F||_{L^\infty(E_B)} \lesssim r(B) ||g||_{L^\infty(B)}. \tag{3.27}$$

Proof: let B be a ball and $g \in L^\infty_0(B)$. Consider

$$\mathcal{S} = \{ V \in L^1(E_B) : \exists f \in L^1(\Gamma) \text{ supported in } \overset{\circ}{B},\ V = df \text{ in } E_B \}.$$

We consider $\mathcal{S}$ as subspace of $L^1(E_B)$ equipped with the norm

$$||V||_{L^1(E_B)} = \sum_{\gamma \in E_B} |V(\gamma)| \mu_\gamma$$

(see Section 2.1.1). Define a linear functional on $\mathcal{S}$ by

$$L(V) := \sum_{x\in B} g(x)f(x)m(x) \text{ if } V = df \in \mathcal{S}.$$

Observe that L is well defined since $\sum\limits_{x\in B} g(x)m(x) = 0$ and it is plain to see that, if $df_1 = df_2$ in E_B, then $f_1 - f_2$ is constant on B. From (P_1) and using the support condition on f, we derive

$$\begin{aligned}
|L(V)| &\leq \sum_{x\in B} |g(x)|\,|f(x)-f_B|m(x)\\
&\leq Cr(B)||g||_{L^\infty(B)} \sum_{x\in B} \nabla f(x)m(x)\\
&\leq Cr(B)\,\|g\|_{L^\infty(B)} \sum_{x\in B}\left(\sum_{y\sim x} p(x,y)\,|f(y)-f(x)|\right) m(x)\\
&= Cr(B)\,\|g\|_{L^\infty(B)} \sum_{x\in B}\sum_{y\sim x} |f(y)-f(x)|\,\mu_{xy}\\
&= Cr(B)\,\|g\|_{L^\infty(B)} \sum_{x\sim y,\ x\in B,\ y\in B} |f(y)-f(x)|\,\mu_{xy}\\
&= Cr(B)||g||_{L^\infty(B)}||V||_{L^1(E_B)}.
\end{aligned}$$

The Hahn-Banach theorem shows that L can be extended to a bounded linear functional on $L^1(E_B)$ with norm not greater than $Cr(B)||g||_\infty$. Thus, there exists $F \in L^\infty(E_B)$ such that, for all $V \in L^1(E_B)$,

$$L(V) = \sum_{\gamma\in E_B} F(\gamma)V(\gamma)\mu_\gamma.$$

In particular, for all $f \in L^1(B)$ vanishing on ∂B, (2.10) yields[2]

$$\sum_{x\in B} g(x)f(x)m(x) = L(df) = \sum_{\gamma\in E_B} F(\gamma)df(\gamma)\mu_\gamma = -\sum_{x\in \overset{\circ}{B}} \delta F(x)f(x)m(x),$$

which ensures that $-\delta F = g$ in $\overset{\circ}{B}$ with

$$\|F\|_{L^\infty} \leq Cr(B)||g||_\infty.$$

□

A consequence of Proposition 3.3 , which will also be useful in the proof of Theorem 2.9, is:

Proposition 3.4 *For all functions f on Γ:*

1. $$\mathcal{M}^+ f \sim Nf,$$

2. $$\nabla f \lesssim Nf.$$

[2]Observe that F and df are square integrable on E_B since E_B is a finite set.

Proof of $\mathcal{M}^+ f \lesssim Nf$: let $x \in \Gamma$. Take F as in the definition of $\mathcal{M}^+ f$, associated to a ball B containing x. Then (2.10), applied with the function f equal to 1 in $\overset{\circ}{B}$ and to 0 on ∂B, shows that $\sum_{y \in \overset{\circ}{B}} (\delta F)(y) m(y) = 0$ so we can write

$$\left| \sum_{y \in \overset{\circ}{B}} f(y)(\delta F)(y)m(y) \right| = \left| \sum_{y \in \overset{\circ}{B}} (f(y) - f_B)(\delta F)(y)m(y) \right|.$$

Thus, (2.16) yields

$$\begin{aligned} \left| \sum_{y \in \overset{\circ}{B}} f(y)(\delta F)(y)m(y) \right| &\leq \frac{1}{r(B)V(B)} \sum_{y \in \overset{\circ}{B}} |f(y) - f_B| \, m(y) \\ &\lesssim Nf(x). \end{aligned}$$

Taking the supremum over all such F, we get

$$\mathcal{M}^+ f(x) \lesssim Nf(x).$$

Proof of $Nf \lesssim \mathcal{M}^+ f$: let $x \in \Gamma$ and $B = B(x_B, r(B))$ a ball containing x. We may and do assume that $r_B \geq 1$, otherwise

$$\sum_{y \in B} |f(y) - f_B| \, m(y) = 0.$$

Define $\widetilde{B} := B(x_B, r(B) + 1)$, so that $B \subset \overset{\circ}{\widetilde{B}}$. If $g \in L^\infty_0(B)$ with $||g||_\infty \leq 1$, extend g by 0 outside B and solve $\delta F = g$ in $\overset{\circ}{\widetilde{B}}$ with $F \in L^\infty \left(E_{\widetilde{B}} \right)$ satisfying (3.27). Extend F by 0 outside $E_{\widetilde{B}}$. Then, setting

$$\widetilde{F} := \frac{F}{Cr\left(\widetilde{B}\right) V(\widetilde{B})},$$

one has

$$\begin{aligned} \frac{1}{r(B)V(B)} \left| \sum_{y \in B} f(y)g(y)m(y) \right| &= \frac{1}{r(B)V(B)} \left| \sum_{y \in \overset{\circ}{\widetilde{B}}} f(y)g(y)m(y) \right| \\ &= \frac{1}{r(B)V(B)} \left| \sum_{y \in \overset{\circ}{\widetilde{B}}} f(y)(\delta F)(y)m(y) \right| \\ &= C \frac{r(\widetilde{B})V(\widetilde{B})}{r(B)V(B)} \left| \sum_{y \in \overset{\circ}{\widetilde{B}}} f(y)(\delta \widetilde{F})(y)m(y) \right| \\ &\leq C\mathcal{M}^+ f(x), \end{aligned}$$

where the last line follows from (D) and the fact that $\widetilde{F}$ satisfies (2.16) . Taking the supremum on the left hand side over all balls containing x, we get $Nf(x) \leq C\mathcal{M}^+ f(x)$. This inequality concludes the proof of 1.

Proof of $\nabla f \lesssim Nf$: let $x \in \Gamma$. Fix $y \sim x$, set $B := B(x,2)$ and define the function F on E in the following way: $F(x,y) = \frac{1}{m(x)}$, $F(y,x) = -\frac{1}{m(x)}$ and $F(u,v) = 0$ whenever $(u,v) \neq (x,y)$ and $(u,v) \neq (y,x)$. Notice that δF is supported in $\overset{\circ}{B}$ and

$$\|F\|_{L^\infty(E_B)} \lesssim \frac{1}{V(B)} \text{ and } \|\delta F\|_{L^\infty(\overset{\circ}{B})} \lesssim \frac{1}{r(B)V(B)}.$$

This and item 1 of Proposition 3.4 yield

$$|\langle df, F\rangle| = |\langle f, \delta F\rangle| \lesssim \mathcal{M}^+ f(x) \lesssim Nf(x).$$

But

$$\langle df, F\rangle = 2(f(y) - f(x))\frac{\mu_{xy}}{m(x)} = 2p(x,y)(f(y) - f(x)),$$

which shows that

$$p(x,y)\,|f(y) - f(x)| \lesssim Nf(x)$$

for all $y \sim x$. The definition of ∇f then yields the desired result. □

Proof of Theorem 2.9: we write it for homogenous spaces, the inhomogeneous case being an immediate consequence. First, assertion 1 in Proposition 3.4 gives at once that $\dot{E}^{1,p}(\Gamma) = \dot{S}^{1,p}(\Gamma)$.

Assume now that $f \in \dot{W}^{1,p}(\Gamma)$ and let $x \in \Gamma$. Since (P_p) holds, there exists $q < p$ such that (P_q) is still valid (see Remark 2.5). For all balls $B \ni x$, (P_q) yields

$$\frac{1}{V(B)} \sum_{y \in B} |f(y) - f_B|\, m(y) \leq Cr(B) \left(\frac{1}{V(B)} \sum_{y \in B} |\nabla f(y)|^q\, m(y) \right)^{\frac{1}{q}},$$

so that, taking the supremum over B,

$$Nf(x) \leq C \left(\mathcal{M}_{HL} |\nabla f|^q\right)^{\frac{1}{q}}(x),$$

where $\mathcal{M}_{HL}$ stands for the Hardy-Littlewood maximal function, given by

$$\mathcal{M}_{HL} f(x) := \sup_{B \ni x} \frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y),$$

where, again, the supremum is taken over all balls B containing x. Since $\nabla f \in L^p(\Gamma)$ and $\mathcal{M}_{HL}$ is $L^{\frac{p}{q}}(\Gamma)$-bounded (this is because (D) holds and $\frac{p}{q} > 1$), one has

$$\left(\sum_{x \in \Gamma} |Nf(x)|^p\, m(x) \right)^{\frac{1}{p}} \leq C \left\|(\mathcal{M}_{HL} |\nabla f|^q)\right\|^{\frac{1}{q}}_{L^{\frac{p}{q}}(\Gamma)} \leq C \left\|\nabla f\right\|_{L^p(\Gamma)},$$

which shows that $Nf \in L^p(\Gamma)$. One therefore has $f \in \dot{S}^{1,p}(\Gamma)$ and $\|f\|_{\dot{S}^{1,p}(\Gamma)} \leq C \|f\|_{\dot{W}^{1,p}(\Gamma)}$.
Take now $f \in \dot{S}^{1,p}(\Gamma)$. Since $Nf \in L^p(\Gamma)$, Proposition 3.2 shows that $f \in \dot{M}^{1,p}(\Gamma)$ and $\|f\|_{\dot{M}^{1,p}(\Gamma)} \leq C \|f\|_{\dot{S}^{1,p}(\Gamma)}$.
Assume finally that $f \in \dot{M}^{1,p}(\Gamma)$ and let $g \in L^p(\Gamma)$ given by (2.12) and satisfying $\|g\|_{L^p(\Gamma)} \leq 2\|f\|_{\dot{M}^{1,p}(\Gamma)}$. Define, for all $x \in \Gamma$, $h(x) := \sum_{y\sim x} (g(y) + g(x))$. Then $h \in L^p(\Gamma)$ and $\|h\|_{L^p(\Gamma)} \leq C\|g\|_{L^p(\Gamma)}$. Indeed, observing that, whenever $x \sim y$, $m(x) \leq Cm(y)$ (this is an immediate consequence of (D)), and using the fact that any point in Γ has at most N neighbours, one obtains

$$\begin{aligned}\sum_{x\in\Gamma} h(x)^p m(x) &\leq C\sum_{x\sim y} (g(x)^p + g(y)^p)\, m(x)\\ &\leq C\sum_{x\in\Gamma} g(x)^p m(x) + C\sum_{y\in\Gamma} g(y)^p m(y)\\ &= C\|g\|^p_{L^p(\Gamma)}.\end{aligned}$$

Now, let $x \in \Gamma$. By (2.12) and the fact that $0 \leq p(x,y) \leq 1$ for all $x, y \in \Gamma$,

$$\nabla f(x) \leq C\sum_{y\sim x} |f(y) - f(x)| \leq C\sum_{y\sim x} (g(x) + g(y)) = Ch(x),$$

so that $\nabla f \in L^p(\Gamma)$ and $\|f\|_{\dot{W}^{1,p}(\Gamma)} \leq C\|f\|_{\dot{M}^{1,p}(\Gamma)}$. This completes the proof. □

4 The Calderón-Zygmund decomposition for Hardy-Sobolev spaces

The present section is devoted to the proof of the Calderón-Zygmund decomposition for Hardy-Sobolev spaces on graphs. The corresponding decomposition on Riemannian manifolds was established in [BD10]. Recall that analogous Calderón-Zygmund decompositions for classical Sobolev spaces were proved in [AC05] on Riemannian manifolds and [BR09] on graphs.

Proposition 4.1 *[Calderón-Zygmund decomposition for Hardy-Sobolev spaces] Let Γ satisfy (D) and (P_1). Let $f \in \dot{S}^{1,1}(\Gamma)$, $\frac{s}{s+1} < q < 1$ and $\alpha > 0$. Then one can find a collection of balls $\{B_i\}_{i\in I}$, functions $b_i \in W^{1,1}(\Gamma)$ and a function $g \in \dot{W}^{1,\infty}(\Gamma)$ such that the following properties hold:*

$$f = g + \sum_i b_i,$$

$$|\nabla g(x)| \leq C\alpha \text{ for all } x \in \Gamma, \tag{4.28}$$

$$\text{supp } b_i \subset B_i, \|b_i\|_1 \leq C\alpha r_i V(B_i), \quad \|\nabla b_i\|_q \leq C\alpha V(B_i)^{1/q} \tag{4.29}$$

$$\sum_i V(B_i) \leq \frac{C}{\alpha}\sum_{x\in B_i} (Nf)(x)m(x) \tag{4.30}$$

and

$$\sum_i \chi_{B_i} \leq K, \tag{4.31}$$

where, for all i, r_i is the radius of B_i, and C and K only depend on q, p and on the constants in (D) and (P_1).

Proof: the proof of Proposition 4.1 follows the main lines of the one of Proposition 3.3 in [BD10], with adaptations due to the discrete context.
Let $f \in \dot{S}^{1,1}(\Gamma)$ and $\alpha > 0$. Define

$$\Omega := \left\{ x \in \Gamma;\ \mathcal{M}_{HL,q}(Nf)(x) > \frac{\alpha}{C} \right\},$$

where C is the implicit constant in item 2 of Proposition 3.4 and $\mathcal{M}_{HL,q}$ is defined by

$$\mathcal{M}_{HL,q}(g)(y) := \left(\mathcal{M}_{HL}\,|g|^q)\right)^{1/q}. \tag{4.32}$$

Let $F := \Gamma \setminus \Omega$.
A consequence of item 2 in Proposition 3.4 is that

$$\nabla f(x) \leq CNf(x) \leq C\mathcal{M}_{HL,q}(Nf)(x) \leq \alpha \text{ for all } x \in F. \tag{4.33}$$

If $\Omega = \emptyset$, then set

$$f = g \text{ and } b_i = 0 \text{ for all } i,$$

so that (4.28) is satisfied by (4.33), and all the other required properties are clearly satisfied.
From now on, assume that $\Omega \neq \emptyset$. First,

$$\begin{aligned} m(\Omega) &\leq \frac{C}{\alpha} \sum_{x\in\Gamma} \mathcal{M}_{HL,q}(Nf)(x) m(x) \\ &= \frac{C}{\alpha} \sum_{x\in\Gamma} (\mathcal{M}_{HL}(|Nf|^q)(x))^{\frac{1}{q}}\, m(x) \\ &\leq \frac{C}{\alpha} \sum_{x\in\Gamma} Nf(x) m(x) < \infty, \end{aligned} \tag{4.34}$$

where, in the last line, we used the fact the $\mathcal{M}_{HL}$ is $L^{1/q}(\Gamma)$-bounded since $q < 1$ and $Nf \in L^1(\Gamma)$. In particular $\Omega \neq \Gamma$ as $m(\Gamma) = +\infty$ (see Remark 2.2).
Definition of the balls B_i: since Ω is a strict subset of Γ, let $\{\underline{B_i}\}_i$ be a Whitney decomposition of Ω (see [CW77]). More precisely, the $\underline{B_i}$ are pairwise disjoint, and there exist two constants $C_2 > C_1 > 1$, only depending on the metric, such that

- $\Omega = \cup_i B_i$ with $B_i = C_1 \underline{B_i}$, and the balls B_i have the bounded overlap property,
- $r_i = r(B_i) = \frac{1}{2} d(x_i, F)$ where x_i is the center of B_i,
- each ball $\overline{B_i} = C_2 B_i$ intersects F (one can take $C_2 = 4C_1$).

For $x \in \Omega$, define $I_x := \{i : x \in B_i\}$. As already seen in [BR09], there exists K such that $\sharp I_x \leq K$, and moreover, for all $i, k \in I_x$, $\frac{1}{3} r_i \leq r_k \leq 3r_i$ and $B_i \subset 7B_k$. The bounded overlap property yields (4.31) and implies

$$\sum_i V(B_i) \lesssim m(\Omega). \tag{4.35}$$

Then, (4.30) follows from (4.31) and (4.34).
The following observation will be used several times: for all i,

$$\left(\frac{1}{V(C_2B_i)}\sum_{x\in C_2B_i}|Nf(x)|^q m(x)\right)^{\frac{1}{q}} \leq C\alpha V(B_i). \tag{4.36}$$

Indeed, the left-hand side of (4.36) is bounded by $\mathcal{M}_{HL,q}(Nf)(y)$ for some $y \in C_2B_i \cap F$, which yields the result.

Definition of the functions b_i: following the construction in Section 5 of [BR09], pick up a partition of unity $\{\chi_i\}_i$ of Ω subordinated to the covering $\{B_i\}_i$. Each χ_i is a Lipschitz function supported in B_i with $0 \leq \chi_i \leq 1$, $||\nabla\chi_i||_\infty \leq \frac{C}{r_i}$ and $\sum_{i\in I}\chi_i(x) = \mathbf{1}_\Omega$ for all $x \in \Gamma$. Moreover, $\nabla\chi_i$ is supported in $C_3B_i \subset \Omega$ with $C_3 < 2$. We set $b_i := (f - f_{B_i})\chi_i$, so that $supp\ b_i \subset B_i$.

Estimate of $\|b_i\|_{L^1(\Gamma)}$: the Sobolev-Poincaré inequality (5.43) applied with $g = Nf$ (recall that $Nf \in L^1(\Gamma)$ and the pair (f, Nf) satisfies (2.12) by Proposition 3.2) and $\lambda = C_2$, as well as (4.36), yield

$$\begin{aligned} \|b_i\|_1 &\leq \sum_{x\in B_i}|f(x) - f_{B_i}|m(x) \\ &\leq Cr_i\left(\frac{1}{V(C_2B_i)}\sum_{x\in C_2B_i}|Nf(x)|^q m(x)\right)^{\frac{1}{q}} V(B_i) \\ &\leq Cr_i\alpha V(B_i). \end{aligned} \tag{4.37}$$

Proof of $\nabla b_i \in L^1(\Gamma)$: since

$$\nabla b_i(x) = \nabla\left((f - f_{B_i})\chi_i\right)(x) \leq \left(\max_{y\sim x}\chi_i(y)\right)\nabla f(x) + |f(x) - f_{B_i}|\,\nabla\chi_i(x)$$

and $\chi_i \leq 1$ on Γ, using (4.36) again, one obtains

$$\begin{aligned} ||\nabla b_i||_1 &\leq \sum_{x\in C_3B_i}|f(x) - f_{B_i}||\nabla\chi_i(x)|m(x) + \sum_{x\in C_3B_i}|\nabla f(x)|m(x) \\ &\leq C\alpha V(B_i) + \sum_{x\in C_3B_i}|\nabla f(x)|m(x) < +\infty. \end{aligned} \tag{4.38}$$

Estimate of $\|\nabla b_i\|_{L^q(\Gamma)}$: using item 2 in Proposition 3.4, (5.43) with $g = Nf$ (and Hölder) and (4.36), we obtain:

$$\begin{aligned} ||\nabla b_i||_q^q &\leq C\left(\sum_{x\in C_3B_i}|\nabla f(x)|^q m(x) + \sum_{x\in C_3B_i}|f(x) - f_{B_i}|^q\,|\nabla\chi_i(x)|^q m(x)\right) \\ &\leq C\sum_{x\in C_2B_i}|Nf(x)|^q m(x) + C\frac{C^q}{r_i^q}r_i^q\left(\sum_{x\in C_2B_i}|Nf(x)|^q m(x)\right) \\ &\leq C\alpha^q V(B_i). \end{aligned} \tag{4.39}$$

Thus (4.29) is proved.

Definition of g: set now $g = f - \sum_i b_i$. Since the sum is locally finite on Ω, g is well-defined on Γ and $g = f$ on F.

Estimate of $|\nabla g|$: since $\sum_{i\in I} \chi_i(x) = 1$ for all $x \in \Omega$, one has

$$g = f\chi_F + \sum_{i\in I} f_{B_i}\chi_i$$

where χ_F denotes the characteristic function of F. We will need the following lemma:

Lemma 4.2 *There exists $C > 0$ such that, for all $j \in I$, all $u \in F \cap 4B_j$ and all $v \in B_j$,*

$$|g(u) - g(v)| \leq C\alpha d(u, v).$$

Let us admit the conclusion of Lemma 4.2 and complete the proof of (4.28). It is enough to check that $|g(x) - g(y)| \leq C\alpha$ for all $x \sim y \in \Gamma$. Three situations may occur:

1. Assume first that $x, y \in \Omega$. Let $j \in I$ such that $x \in B_j$. Since $\chi_F(x) = \chi_F(y) = 0$ and $\sum_i \chi_i = 1$ on Γ, it follows that

$$g(y) - g(x) = \sum_{i\in I} \left(f_{B_i} - f_{B_j}\right)(\chi_i(y) - \chi_i(x)),$$

so that $|g(y) - g(x)| \leq C \sum_{i\in I} |f_{B_i} - f_{B_j}| \nabla\chi_i(x) := h(x)$.

We claim that $|h(x)| \leq C\alpha$, which will end the proof in this case. Let $i \in I$ be such that $\nabla\chi_i(x) \neq 0$, so that $d(x, B_i) \leq 1$, hence $r_i \leq 3r_j + 1 \leq 4r_j$ and $B_i \subset 10B_j$. An application of (5.43) with $g = Nf$ and of (4.36) yields

$$\begin{aligned} |f_{B_i} - f_{10B_j}| &\leq \frac{1}{V(B_i)} \sum_{y\in B_i} |f(y) - f_{10B_j}|m(y) \\ &\leq \frac{C}{V(B_j)} \sum_{y\in 10B_j} |f(y) - f_{10B_j}|m(y) \\ &\leq Cr_j \left(\frac{1}{V(10B_j)} \sum_{y\in 10B_j} |Nf(y)|^q m(y)\right)^{1/q} \\ &\leq Cr_j\alpha. \end{aligned} \tag{4.40}$$

Analogously $|f_{10B_j} - f_{B_j}| \leq Cr_j\alpha$. Hence

$$\begin{aligned} |h(x)| &= \left|\sum_{i\in I; x\in 2B_i} (f_{B_i} - f_{B_j})\nabla\chi_i(x)\right| \\ &\leq C \sum_{i\in I; x\in 2B_i} |f_{B_i} - f_{B_j}| r_i^{-1} \\ &\leq CK\alpha. \end{aligned} \tag{4.41}$$

2. Assume now that $x \in \overset{\circ}{F}$, so that $y \in F$. In this case $|g(y) - g(x)| = |f(x) - f(y)| \leq C\nabla f(x) \leq C\alpha$ by (4.33).

3. Assume finally that $x \in \partial F$.

   i. If $y \in F$, as already seen, $|g(y) - g(x)| = |f(x) - f(y)| \leq C\nabla f(x) \leq C\alpha$ by (4.33).

   ii. Assume finally that $y \in \Omega$. There exists $j \in I$ such that $y \in B_j$. Since $x \sim y$, one has $x \in 4B_j$, Lemma 4.2 therefore yields

$$|g(x) - g(y)| \leq C\alpha d(x, y) \leq C\alpha.$$

The case when $x \in \Omega$ and $y \in F$ is contained in Case 3.ii by symmetry, since $y \in \partial F$. Thus the proof of Proposition 4.1 is complete. □

Proof of Lemma 4.2: it is analogous to the one of Lemma 5.1 in [BR09]. The only difference is that one uses (5.43) instead of the Poincaré inequality applied in [BR09]. □

5 Proofs of the characterization of Hardy-Sobolev spaces

We now turn to the proof of Theorem 2.12. Let us explain the strategy. We first establish that $\dot{S}^{1,1}(\Gamma) = \dot{M}^{1,1}(\Gamma)$. The inclusion $\dot{S}^{1,1}(\Gamma) \subset \dot{M}^{1,1}(\Gamma)$ is proved exactly in the same way as the corresponding inclusion in Theorem 2.9. The converse is more involved, since the Hardy-Littlewood maximal function is not $L^1(\Gamma)$-bounded, and the proof relies on a Sobolev-Poincaré inequality.
The identity $\dot{S}^{1,1}(\Gamma) = \dot{HS}^1_{\max}(\Gamma)$ is an immediate consequence of item 1 in Proposition 3.4.
Finally, we check that $\dot{S}^{1,1}(\Gamma) = \dot{HS}^1_{ato}(\Gamma)$, using the Sobolev-Poincaré inequality again, as well as an adapted Calderón-Zygmund decomposition.

5.1 Sharp maximal characterization of $\dot{M}^{1,1}(\Gamma)$

A straightforward consequence of Proposition 3.2 is that $\dot{S}^{1,1}(\Gamma) \subset \dot{M}^{1,1}(\Gamma)$.
The proof of the converse inclusion relies, as the proof of Theorem 3 in [KT07], on a Sobolev-Poincaré inequality ([Haj03b], theorem 8.7) :

Theorem 5.1 *Let $p \in \left[\frac{s}{s+1}, s\right)$[3], $B \subset \Gamma$ be a ball with radius r, $f \in \dot{M}^{1,p}(B)$ and $g \in L^p(B)$ such that (f, g) satisfies* (2.12) *in B (see Remark 2.7). Then (f, g) satisfies the following Sobolev-Poincaré inquality: for all $\lambda > 1$, there is a constant $C > 0$ only depending on the constant in (D) and λ such that*

$$\left(\frac{1}{V(B)} \sum_{x \in B} |f(x) - f_B|^{p^*} m(x) \right)^{\frac{1}{p^*}} \leq Cr \left(\frac{1}{V(\lambda B)} \sum_{x \in \lambda B} g(x)^p m(x) \right)^{\frac{1}{p}} \tag{5.42}$$

where $p^* := \frac{sp}{s-p}$.

[3] where s is given by (2.3).

An easy consequence of Theorem 5.1 is that, for all functions $f \in \dot{M}^{1,1}(\Gamma)$, all $q \in \left[\frac{s}{s+1}, s\right)$, all balls $B \subset \Gamma$ of radius r and all $\lambda > 1$,

$$\frac{1}{V(B)} \sum_{x \in B} |f(x) - f_B| \, m(x) \leq Cr \left(\frac{1}{V(\lambda B)} \sum_{x \in \lambda B} g(x)^q m(x) \right)^{\frac{1}{q}} \tag{5.43}$$

whenever (f, g) satisfies (2.12). Indeed, it is enough to observe that $g \in L^{\frac{s}{s+1}}(\lambda B)$, apply Theorem 5.1 with $p = \frac{s}{s+1}$, since $p^* = 1$ and use Hölder inequality.

Take now $f \in \dot{M}^{1,1}(\Gamma)$, $q \in \left[\frac{s}{s+1}, 1\right)$ and g such that (2.12) and (5.43) hold and $\|g\|_{L^1(\Gamma)} \leq 2 \|f\|_{\dot{M}^{1,1}(\Gamma)}$. The inequality (5.43) yields

$$Nf(y) \lesssim \mathcal{M}_{HL,q} g(y)$$

for all $y \in \Gamma$, where $\mathcal{M}_{HL,q}$ was defined by (4.32). Since $1/q > 1$, the Hardy-Littlewood maximal function is $L^{1/q}(\Gamma)$-bounded, which implies that

$$\|Nf\|_{L^1(\Gamma)} \lesssim \|g\|_{L^1(\Gamma)} \lesssim \|f\|_{\dot{M}^{1,1}(\Gamma)} .$$

This ends the proof of the inclusion $\dot{M}^{1,1}(\Gamma) \subset \dot{S}^{1,1}(\Gamma)$. □

5.2 Maximal characterization

The identity $\dot{S}^{1,1}(\Gamma) = \dot{H}S^1_{\max}(\Gamma)$ is an immediate consequence of item 1 in Proposition 3.4.

5.3 Atomic decomposition

We prove now that $\dot{H}S^1_{t,ato}(\Gamma) = \dot{S}^{1,1}(\Gamma)$ for all $t \in (1, +\infty]$.

5.3.1 $\dot{H}S^1_{t,ato}(\Gamma) \subset \dot{S}^1_1(\Gamma)$

For the proof of this inclusion, we have to clarify the link between convergence in $\dot{H}S^1_{t,ato}(\Gamma)$ and pointwise convergence:

Proposition 5.2 *Let $f \in \dot{H}S^1_{t,ato}(\Gamma)$ and write*

$$f = \sum_j \lambda_j a_j,$$

where $\sum_j |\lambda_j| < +\infty$, for all j, a_j is a homogeneous Hardy-Sobolev $(1,t)$-atom and the series converges in $\dot{W}^{1,1}(\Gamma)$. Then, for all k, there exists $c_k \in \mathbb{R}$ such that, for all $x \in \Gamma$,

$$f(x) = \lim_{k \to +\infty} \sum_{j=0}^{k} \lambda_j a_j(x) - c_k.$$

The proof follows from:

Lemma 5.3 *Let $(h_k)_{k\geq 1}\in \dot{W}^{1,1}(\Gamma)$. If $\lim_{k\to+\infty}\|\nabla h_k\|_{L^1(\Gamma)}=0$, then, for all $k\geq 1$, there exists $c_k\in\mathbb{R}$ such that*

$$\lim_{k\to+\infty} h_k(x)-c_k=0.$$

Proof of Lemma 5.3: assume first that there exists $x_0\in\Gamma$ such that $h_k(x_0)=0$ for all $k\geq 1$. Then, for all $x\in\Gamma$, $\lim_{k\to+\infty}h_k(x)=0$. Indeed, the very definition of ∇h_k implies that, for all $x,y\in\Gamma$ with $x\sim y$, $\lim_{k\to+\infty}(h_k(x)-h_k(y))=0$. The conclusion then readily follows for all $j\geq 1$ and for all $x\in B(x_0,j)$ by induction on j.
In the general case, fix $x_0\in\Gamma$ and define $g_k(x):=h_k(x)-h_k(x_0)$ for all $k\geq 1$ and all $x\in\Gamma$. What we have just seen means that $\lim_{k\to+\infty}g_k(x)=0$, which yields the desired conclusion with $c_k:=h_k(x_0)$. □

Proof of Proposition 5.2: it is an immediate consequence of Lemma 5.3 applied with $h_k:=f-\sum_{j=0}^k\lambda_j a_j$. □

Proposition 5.4 *Assume that Γ satisfies (D) and (P_1). Let $t\in(1,+\infty]$.*

1. *Let a be a homogeneous $(1,t)$ atom. Then $a\in\dot{S}_1^1(\Gamma)$ with $||a||_{S_1^1}\leq C$.*

2. *One has $\dot{H}S^1_{t,ato}(\Gamma)\subset\dot{S}^1_1(\Gamma)$ and there exists $C>0$ such that, for all $f\in\dot{H}S^1_{t,ato}(\Gamma)$,*

$$||f||_{\dot{S}^1_1(\Gamma)}\leq C||f||_{\dot{H}S^1_{ato}(\Gamma)}.$$

Proof: for 1, let a be a homogeneous $(1,t)$ atom supported in ball $B=B(x,r)$. We want to prove that $Na\in L^1(\Gamma)$ and that $\|Na\|_{L^1(\Gamma)}\leq C$. For all $y\in\Gamma$, and all balls $B'\ni y$, (P_1) yields:

$$\begin{aligned}\frac{1}{r(B')V(B')}\sum_{z\in B'}|a(z)-a_{B'}|m(z) &\leq \frac{C}{V(B')}\sum_{z\in B'}\nabla a(z)m(z)\\ &\leq \mathcal{M}_{HL}(\nabla a)(y),\end{aligned}$$

so that

$$Na(y)\lesssim\mathcal{M}_{HL}(\nabla a)(y). \tag{5.44}$$

As a consequence,

$$\begin{aligned}\sum_{y\in B(x,4r)}Na(y)m(y) &\leq CV(x,4r)^{1/t'}\left(\sum_{y\in B(x,4r)}\left(\mathcal{M}_{HL}(\nabla a)(y)\right)^t m(y)\right)^{1/t}\\ &\leq CV(x,4r)^{1/t'}\|\nabla a\|_{L^t(\Gamma)}\\ &\leq C,\end{aligned} \tag{5.45}$$

where the first line follows from Hölder and (5.44), the second one from the L^t-boundedness of the Hardy-Littlewood maximal function and the last one from the doubling property and the second item in Definition 2.11.

Let $k \geq 2$ and $y \in B(x, 2^{k+1}r) \setminus B(x, 2^k r)$. Consider an arbitrary ball B' containing y. One has

$$\begin{aligned}\frac{1}{r(B')V(B')} \sum_{z \in B'} |a(z) - a_{B'}| \, m(y) &= \frac{1}{r(B')V(B')} \sum_{z \in B' \cap B} |a(z) - a_{B'}| \, m(z) \\ &+ \frac{1}{r(B')V(B')} \sum_{z \in B' \setminus B} |a_{B'}| \, m(z) \\ &\leq \frac{3}{r(B')V(B')} \sum_{z \in B' \cap B} |a(z)| \, m(z).\end{aligned}$$

It is easily checked that, if $B' \cap B \neq \emptyset$, then $r(B') > 2^{k-1}r$ and (D) yields $V(x, 2^{k+1}r) \leq CV(B')$. As a consequence of this observation and (P_1) (remember that $a_B = 0$),

$$\begin{aligned}Na(y) &\leq \frac{C}{2^{k-1}rV(2^{k+1}B)} \sum_{z \in B} |a(z)| \, m(z) \\ &\leq \frac{C}{2^{k-1}V(2^{k+1}B)} \sum_{z \in B} |\nabla a(z)| \, m(z) \\ &\leq \frac{C}{2^{k-1}V(2^{k+1}B)}.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{y \notin B(x,4r)} Na(y)m(y) &= \sum_{k \geq 2} \sum_{y \in B(x,2^{k+1}r) \setminus B(x,2^k r)} Na(y)m(y) \\ &\leq \sum_{k \geq 2} \frac{C}{2^{k-1}V(2^{k+1}B)} V(2^{k+1}B) \\ &\leq C.\end{aligned} \tag{5.46}$$

Gathering (5.45) and (5.46), one obtains $\|Na\|_{L^1(\Gamma)} \leq C$.
Now, for assertion 2 in Proposition 5.4, if $f \in \dot{H}S^1_{t,ato}(\Gamma)$, take an atomic decomposition of f : $f = \sum_i \lambda_i a_i$ where each a_i is an atom and $\sum_i |\lambda_i| \leq 2 \|f\|_{\dot{H}S^1_{t,ato}(\Gamma)}$. By Proposition 5.2, pick up a sequence $(c_k)_{k \geq 1} \in \mathbb{R}$ such that, for all $x \in \Gamma$,

$$f(x) = \lim_{k \to +\infty} \sum_{j=0}^{k} \lambda_j a_j(x) - c_k = \lim_{k \to +\infty} f_k(x) - c_k$$

where, for all k, $f_k := \sum_{j=0}^k \lambda_j a_j$.
Let $x \in \Gamma$ and B be a ball containing x. Observe that

$$f_B = \frac{1}{V(B)} \sum_{y \in B} f(y)m(y) = \lim_{k \to +\infty} \frac{1}{V(B)} \sum_{y \in B} (f_k(y) - c_k) \, m(y) = \lim_{k \to +\infty} ((f_k)_B - c_k) \, .$$

As a consequence,

$$\frac{1}{V(B)} \sum_{y \in B} |f(y) - f_B| \, m(y) = \lim_{k \to +\infty} \frac{1}{V(B)} \sum_{y \in B} |f_k(y) - (f_k)_B| \, m(y).$$

For all $k \geq 1$,

$$\sum_{y\in B} |f_k(y) - (f_k)_B|\, m(y) \leq \sum_{j=0}^{k} |\lambda_j| \sum_{y\in B} |a_j(y) - (a_j)_B|\, m(y),$$

so that

$$\frac{1}{r(B)V(B)} \sum_{y\in B} |f(y) - f_B|\, m(y) \leq \sum_{j=0}^{+\infty} |\lambda_j|\, Na_j(x).$$

Since $\|Na_j\|_{L^1(\Gamma)} \leq C$ and $\sum_j |\lambda_j| \leq 2\,\|f\|_{\dot{H}S^1_{t,ato}(\Gamma)}$, Proposition 5.4 is proved. $\square$

Remark 5.5 *Observe that, in the above argument, if condition* 3 *in Definition 2.11 is replaced by condition* $3'$ *in Remark 2.13, then the previous computation is still valid, since one has, using Hölder,*

$$\begin{aligned} Na(y) &\leq \frac{C}{2^{k-1} r V(2^{k+1}B)} \sum_{z\in B} |a(z)|\, m(z) \\ &\leq \frac{C}{2^{k-1} r V(2^{k+1}B)} \|a\|_{L^t(B)}\, V(B)^{1/t'} \\ &\leq \frac{C}{2^{k-1} V(2^{k+1}B)}. \end{aligned}$$

5.3.2 $\dot{S}^1_1(\Gamma) \subset \dot{H}S^1_{q^*,ato}(\Gamma)$

The proof of the inclusion $\dot{S}^{1,1}(\Gamma) \subset \dot{H}S^1_{ato}(\Gamma)$ relies on the Calderón-Zygmund decomposition for functions in $\dot{S}^{1,1}(\Gamma)$ given by Proposition 4.1:

Proposition 5.6 *Let* Γ *satisfying* (D) *and* (P_1). *Let* $f \in \dot{S}^{1,1}(\Gamma)$. *Then for all* $\frac{s}{s+1} < q < 1, q^* = \frac{sq}{s-q}$, *there is a sequence of* $(1, q^*)$ *Hardy-Sobolev atoms* $\{a_j\}_j$, *and a sequence of scalars* $\{\lambda_j\}_j \in l^1$ *such that*

$$f = \sum_j \lambda_j a_j \quad \text{in } \dot{W}^{1,1}(\Gamma), \quad \text{and} \quad \sum |\lambda_j| \leq C_q ||f||_{\dot{S}^{1,1}(\Gamma)}.$$

Consequently, $\dot{S}^{1,1}(\Gamma) \subset \dot{H}S^1_{q^*,ato}(\Gamma)$ *with* $||f||_{\dot{H}S^1_{q^*,ato}(\Gamma)} \leq C_q ||f||_{\dot{S}^{1,1}(\Gamma)}$.

Proof: the proof is analogous to the one of Proposition 3.4 in [BD10], which deals with the case of Riemannian manifolds, and is also inspired by the proof of the atomic decomposition for Hardy spaces in [Ste93], section III.2.3. We may and do assume that f is not constant on Γ, otherwise one can take $a_j = 0$ for all j.
Let $f \in \dot{S}^{1,1}(\Gamma)$. For every $j \in \mathbb{Z}^*$, we take the Calderón-Zygmund decomposition for f at level $\alpha = 2^j$ given by Proposition 4.1. Then

$$f = g^j + \sum_i b_i^j$$

with b_i^j, g^j satisfying the properties of Proposition 4.1. We first claim

$$f = \sum_{-\infty}^{\infty} (g^{j+1} - g^j), \tag{5.47}$$

where the series converges in $\dot{W}^{1,1}(\Gamma)$.
To see this, observe first that $g^j \to f$ in $\dot{W}^{1,1}(\Gamma)$ as $j \to +\infty$. Indeed, since the sum is locally finite we can write, using (4.38), (4.35) and the facts that $C_3 B_i^j \subset \Omega$ and that the $C_3 B_i^j$ have the bounded overlap property,

$$\begin{aligned} \left\| \nabla (g^j - f) \right\|_{L^1(\Gamma)} &= \left\| \nabla \left(\sum_i b_i^j \right) \right\|_{L^1(\Gamma)} \leq \sum_i \| \nabla b_i^j \|_{L^1(\Gamma)} \\ &\leq C 2^j m(\Omega^j) + C \sum_{x \in \Omega_j} |\nabla f(x)| m(x) \\ &:= I_j + II_j, \end{aligned} \tag{5.48}$$

where $\Omega^j := \left\{ x \in \Gamma, \mathcal{M}_{HL,q}(Nf)(x) > \frac{2^j}{C} \right\}$. Observe that $\Omega^{j+1} \subset \Omega^j$ for all $j \in \mathbb{Z}$.
Observe that

$$\sum_{j \in \mathbb{Z}} 2^j m(\Omega^j) \lesssim \int_0^{+\infty} m\left(\{ x \in \Gamma;\ \mathcal{M}_{HL,q}(Nf)(x) > t \} \right) dt = \| \mathcal{M}_{HL,q}(Nf) \|_{L^1(\Gamma)} < +\infty. \tag{5.49}$$

This implies that, when $j \to +\infty$, $I_j \to 0$. Since $\nabla f \in L^1(\Gamma)$ and $m(\Omega_j) \to 0$ when $j \to +\infty$, one has $II_j \to 0$ when $j \to +\infty$. Thus, (5.48) shows that

$$\lim_{j \to +\infty} g^j = f \text{ in } \dot{W}^{1,1}(\Gamma).$$

Next, when $j \to -\infty$, we want to show $\| \nabla g_j \|_{L^1(\Gamma)} \to 0$. If $F^j := \Gamma \setminus \Omega^j$, an immediate consequence of Remark 2.8 is that, since f is not constant on Γ,

$$\bigcap_{j \in \mathbb{Z}} F^j = \emptyset. \tag{5.50}$$

Write

$$\begin{aligned} \left\| \nabla g^j \right\|_{L^1(\Gamma)} &\lesssim \sum_{x \sim y,\ x, y \in F^j} \left| g^j(x) - g^j(y) \right| m(x) \\ &+ \sum_{x \sim y,\ x, y \in \Omega^j} \left| g^j(x) - g^j(y) \right| m(x) \\ &+ \sum_{x \sim y,\ x \in F^j,\ y \in \Omega^j} \left| g^j(x) - g^j(y) \right| m(x) \\ &:= A_j + B_j + C_j. \end{aligned}$$

If $x \sim y$ with $x \in F^j$ and $y \in F^j$, $|g^j(x) - gj(y)| = |f(x) - f(y)|$, so that

$$A_j \lesssim \sum_{x \in F^j} \nabla f(x) m(x),$$

which implies that $A_j \to 0$ when $j \to -\infty$, since $\nabla f \in L^1(\Gamma)$ and (5.50) holds.
Moreover,

$$B_j \lesssim \sum_{x\in\Omega_j} \nabla g^j(x)m(x) \lesssim 2^j m(\Omega^j),$$

and this quantity goes to 0 when $j \to -\infty$ by (5.49).
Finally, if $x \sim y$ with $x \in F^j$ and $y \in \Omega^j$, $|g^j(x) - g^j(y)| \lesssim \nabla g^j(y)$ and, since $m(x)$ and $m(y)$ are comparable when $x \sim y$, one has

$$C_j \lesssim \sum_{y\in\Omega^j} \nabla g^j(y)m(y)$$

which goes to 0 when $j \to -\infty$. This ends the proof of (5.47).

Introduce a partition of unity $(\chi_k^j)_k$ subordinated to balls B_k^j corresponding to Ω^j as in the proof of Proposition 4.1. We will need two observations:

Lemma 5.7 1. *For all j, k, l, if there exist $x \in B_k^j$ and $y \in B_l^{j+1}$ with $x \sim y$, then*

$$r_l^{j+1} \leq 4r_k^j. \tag{5.51}$$

2. *There exists $C > 0$ such that, for all j,*

$$\sum_k \mathbf{1}_{2B_k^j} \leq C. \tag{5.52}$$

We postpone the proof of Lemma 5.7 and end up the proof of the atomic decomposition of f. Set $g^{j+1} - g^j := l^j$ and decompose l_j as $l^j = \sum\limits_k l_k^j$ with

$$l_k^j := (f - d_k^j)\chi_k^j - \sum_l (f - d_l^{j+1})\chi_l^{j+1}\chi_k^j + \sum_l c_{k,l}^j \chi_l^{j+1}, \tag{5.53}$$

where, for all j, k,

$$d_k^j := \frac{1}{\sum_y \chi_k^j(y)m(y)} \sum_y f(y)\chi_k^j(y)m(y),$$

and

$$c_{k,l}^j := \frac{1}{\sum_{y\in B_l^{j+1}} \chi_l^{j+1}(y)m(y)} \sum_{x\in B_l^{j+1}} \left(f(x) - d_l^{j+1}\right) \chi_l^{j+1}(x)\chi_k^j(x)m(x).$$

First, the identity $l^j = \sum\limits_k l_k^j$ holds by definition of g^j and g^{j+1} and since $\sum\limits_k \chi_k^j = 1$ on the support of χ_l^{j+1} and, for all l, $\sum\limits_k c_{k,l}^j = 0$.

We now claim that, up to a constant, $2^{-j}V(B_k^j)^{-1}l_k^j$ is a homogeneous Hardy-Sobolev $(1, q^*)$ atom. Indeed, the cancellation condition

$$\sum_{x\in\Gamma} l_k^j(x)m(x) = 0$$

for all k follows from the fact that $\sum_{x\in\Gamma}\left(f(x)-d_k^j\right)\chi_k^j(x)m(x)=0$ and the definition of $c_{k,l}^j$, which immediately gives, for all l, $\sum_{x\in\Gamma}\left(\left(f(x)-d_l^{j+1}\right)\chi_l^{j+1}(x)\chi_k^j(x)-c_{k,l}^j\chi_l^{j+1}(x)\right)m(x)=0$. A consequence of (5.51) is that l_k^j is supported in the ball $9B_k^j$, therefore ∇l_k^j is supported in $18B_k^j$.

Let us now prove that

$$\left\|\nabla l_k^j\right\|_{L^{q^*}(\Gamma)}\lesssim 2^j V(B_k^j)^{1/q^*}. \tag{5.54}$$

Let $x,y\in\Gamma$ such that $x\sim y$. Write

$$\begin{array}{rcl} l_k^j(y)-l_k^j(x) & = & \left((f(y)-f(x))\chi_k^j(y)-\sum_l (f(y)-f(x))\chi_l^{j+1}(y)\chi_k^j(y)\right)\\ & + & \left(f(x)-d_k^j\right)\left(\chi_k^j(y)-\chi_k^j(x)\right)\\ & - & \sum_l\left(f(x)-d_l^{j+1}\right)\left(\chi_l^{j+1}(y)\chi_k^j(y)-\chi_l^{j+1}(x)\chi_k^j(x)\right)\\ & + & \sum_l c_{k,l}\left(\chi_l^{j+1}(y)-\chi_l^{j+1}(x)\right)\\ & := & \Delta_1(x,y)+\Delta_2(x,y)+\Delta_3(x,y)+\Delta_4(x,y). \end{array} \tag{5.55}$$

Let us estimate $\Delta_i(x,y)$ for $1\le i\le 4$.

Estimate of Δ_1: compute

$$\Delta_1(x,y)=(f(y)-f(x))\chi_k^j(y)\left(1-\mathbf{1}_{\Omega^{j+1}}(y)\right).$$

As a consequence, if $\Delta_1(x,y)\neq 0$, one has $y\in B_k^j\cap(\Omega^j\setminus\Omega^{j+1})$, so that $x\in 2B_k^j$. By item 2 in Proposition 3.4, one has $\nabla f(y)\le C2^j$, so that $|f(y)-f(x)|\le C2^j$. As a consequence, for all $x\in\Gamma$,

$$\sum_{y\sim x}\left|\Delta_1(x,y)\right|^{q^*}\le C2^{jq^*}.$$

Therefore, by (D),

$$\sum_{x\in 2B_k^j}\sum_{y\sim x}\left|\Delta_1(x,y)\right|^{q^*}m(x)\le C2^{jq^*}V(B_k^j). \tag{5.56}$$

Estimate of Δ_2: observe first that if $\Delta_2(x,y)\neq 0$, then $y\in B_k^j$ or $x\in B_k^j$, so that $x\in 2B_k^j$. Since $\nabla\chi_k^j\le\frac{C}{r_k^j}$ on Γ, one has, for all $x\in\Gamma$,

$$\sum_{y\sim x}\left|\Delta_2(x,y)\right|^{q^*}\le\frac{C}{\left(r_k^j\right)^{q^*}}\left|f(x)-d_k^j\right|^{q^*}.$$

As a consequence,

$$\sum_{x\in 2B_k^j}\sum_{y\sim x}\left|\Delta_2(x,y)\right|^{q^*}m(x)\le\frac{C}{\left(r_k^j\right)^{q^*}}\sum_{x\in 2B_k^j}\left|f(x)-d_k^j\right|^{q^*}m(x).$$

But

$$\left\|f-d_k^j\right\|_{L^{q^*}(2B_k^j)} \leq \left\|f-f_{B_k^j}\right\|_{L^{q^*}(2B_k^j)} + \left|d_k^j - f_{B_k^j}\right| V^{1/q^*}(2B_k^j),$$

and

$$\begin{aligned} V^{1/q^*}(2B_k^j)\left|d_k^j - f_{B_k^j}\right| &= V^{1/q^*}(2B_k^j)\left|\frac{1}{\sum_y \chi_k^j(y)m(y)}\sum_z \left(f(z)-f_{B_k^j}\right)\chi_k^j(z)m(z)\right| \\ &\leq C\left(\frac{V(B_k^j)}{\sum_y \chi_k^j(y)m(y)}\sum_{z\in B_k^j}\left|f(z)-f_{B_k^j}\right|^{q^*} m(z)\right)^{1/q^*} \\ &\leq C\left(\sum_{z\in B_k^j}\left|f(z)-f_{B_k^j}\right|^{q^*} m(z)\right)^{1/q^*}. \end{aligned}$$

Thus,

$$\sum_{x\in 2B_k^j}\left|f(x)-d_k^j\right|^{q^*} m(x) \leq C\sum_{z\in 2B_k^j}\left|f(z)-f_{B_k^j}\right|^{q^*} m(z). \tag{5.57}$$

Therefore, by Theorem 5.1 and (4.36),

$$\begin{aligned} \sum_{x\in 2B_k^j}\sum_{y\sim x}\left|\Delta_2(x,y)\right|^{q^*} m(x) &\leq \frac{C}{\left(r_k^j\right)^{q^*}}\sum_{x\in 2B_k^j}\left|f(x)-f_{B_k^j}\right|^{q^*} m(x) \\ &\leq CV(B_k^j)\left(\frac{1}{V(4C_2B_k^j)}\sum_{x\in 4C_2B_k^j} Nf(x)^q m(x)\right)^{\frac{q^*}{q}} \\ &\leq CV(B_k^j)2^{jq^*}. \end{aligned}$$

Estimate of $\Delta_3(x,y)$: first,

$$\begin{aligned} -\Delta_3(x,y) &= \sum_l \left(f(x)-f_{B_l^{j+1}}\right)\chi_k^j(y)\left(\chi_l^{j+1}(y)-\chi_l^{j+1}(x)\right) \\ &+ \sum_l \left(f(x)-f_{B_l^{j+1}}\right)\chi_l^{j+1}(x)\left(\chi_k^j(y)-\chi_k^j(x)\right) \\ &= \Delta_3^1(x,y)+\Delta_3^2(x,y). \end{aligned}$$

For $\Delta_3^1(x,y)$, notice that the sum may be computed over the $l\in I^j(x)$, where

$$I^j(x) := \left\{l;\text{ there exists } y\sim x \text{ such that } y\in B_k^j \text{ and } x \text{ or } y \text{ belong to } B_l^{j+1}\right\}.$$

For $l\in I^j(x)$, $x\in 2B_k^j \cap 2B_l^{j+1}$ and $r_l^{j+1}\leq 4r_k^j$ by Lemma 5.7. Since $\left|\chi_l^{j+1}(y)-\chi_l^{j+1}(x)\right| \leq \frac{C}{r_l^{j+1}}$, one has, for all $x\in\Gamma$,

$$\sum_{y\sim x}\left|\Delta_3^1(x,y)\right|^{q^*} \leq \sum_{l\in I^j(x)}\frac{C}{\left(r_l^{j+1}\right)^{q^*}}\left|f(x)-f_{B_l^{j+1}}\right|^{q^*}.$$

Notice that, by item 2 in Lemma 5.7, $\sharp I^j(x) \leq C$. It follows that

$$\begin{aligned}
\sum_{x\in 2B_k^j\cap 2B_l^{j+1}} \sum_{y\sim x} \left|\Delta_3^1(x,y)\right|^{q^*} m(x) &\leq \sum_{x\in 2B_k^j\cap 2B_l^{j+1}} \sum_{l\in I^j(x)} \frac{C}{\left(r_l^{j+1}\right)^{q^*}} \left|f(x)-f_{B_l^{j+1}}\right|^{q^*} m(x) \\
&= C\sum_l \frac{1}{\left(r_l^{j+1}\right)^{q^*}} \sum_{x\in 2B_k^j\cap 2B_l^{j+1},\ l\in I^j(x)} \left|f(x)-f_{B_l^{j+1}}\right|^{q^*} m(x) \\
&\leq C \sum_{l;\ B_l^{j+1}\subset CB_k^j} V(CB_l^{j+1}) \left(\frac{1}{V(4CB_l^{j+1})} \sum_{x\in 4CB_l^{j+1}} Nf(x)^q m(x)\right)^{\frac{q^*}{q}} \\
&\leq C \sum_{l;\ B_l^{j+1}\subset CB_k^j)} V(CB_l^{j+1}) 2^{(j+1)q^*} \\
&\leq CV(CB_k^j)2^{jq^*}.
\end{aligned}$$

In this computation, we used the fact that, for $l \in I^j(x)$, one has $r_l^{j+1} \leq 4r_k^j$ and, since $2B_l^{j+1}\cap 2B_k^j \neq \emptyset$, $B_l^{j+1}\subset CB_k^j$.
For $\Delta_3^2(x,y)$, arguing similarly, the sum may be restricted to the $l\in J^j(x)$ where

$$J^j(x) := \left\{l;\ x\in B_l^{j+1} \text{ and there exists } y\sim x \text{ such that } y\in B_k^j \text{ or } x\in B_k^j\right\}.$$

For $l\in J^j(x)$, $x\in B_l^{j+1}\cap 2B_k^j$ and $r_l^{j+1}\leq 4r_k^j$. Again, $\sharp J^j(x)\leq C$. Arguing as before, one obtains

$$\sum_{y\sim x} \left|\Delta_3^2(x,y)\right|^{q^*} \leq \sum_{l\in J^j(x)} \frac{C}{\left(r_k^j\right)^{q^*}} \left|f(x)-f_{B_l^{j+1}}\right|^{q^*}.$$

As a consequence,

$$\begin{aligned}
\sum_{x\in 2B_k^j\cap B_l^{j+1}} \sum_{y\sim x} \left|\Delta_3^2(x,y)\right|^{q^*} m(x) &\leq \sum_{x\in 2B_k^j\cap B_l^{j+1}} \sum_{l\in J^j(x)} \frac{C}{\left(r_k^j\right)^{q^*}} \left|f(x)-f_{B_l^{j+1}}\right|^{q^*} m(x) \\
&\leq \sum_{x\in 2B_k^j\cap B_l^{j+1}} \sum_{l\in J^j(x)} \frac{C}{\left(r_l^{j+1}\right)^{q^*}} \left|f(x)-f_{B_l^{j+1}}\right|^{q^*} m(x) \\
&\leq C \sum_{l;\ B_l^{j+1}\subset CB_k^j} V(CB_l^{j+1}) \left(\frac{1}{V(4CB_l^{j+1})} \sum_{x\in 4CB_l^{j+1}} Nf(x)^q m(x)\right)^{\frac{q^*}{q}}. \\
&\leq C \sum_{l;\ B_l^{j+1}\subset CB_k^j} V(CB_l^{j+1}) 2^{(j+1)q^*} \\
&\leq CV(CB_k^j)2^{jq^*}.
\end{aligned}$$

Estimate of Δ_4: note first that $c_{k,l}^j = 0$ when $B_k^j\cap B_l^{j+1}=\emptyset$ and $|c_{k,l}^j|\leq C2^j r_l^{j+1}$ thanks to (4.37). As a consequence, $\left|c_{k,l}^j\left(\chi_l^{j+1}(y)-\chi_l^{j+1}(x)\right)\right| \leq C2^j$ for every l. It follows that, for all x,

$$\sum_l \sum_{y\sim x} \left|c_{k,l}^j\right| \left|\chi_l^{j+1}(y)-\chi_l^{j+1}(x)\right| \leq C2^j.$$

Therefore,

$$\sum_{x\in CB_k^j}\sum_{y\sim x}|\Delta_4(x,y)|^{q^*}m(x)\leq C2^{(j+1)q^*}V(B_k^j).$$

Gathering the estimates on Δ_i, $1\leq i\leq 4$, we obtain (5.54).

We now set $a_k^j=C^{-1}2^{-j}V\left(B_k^j\right)^{-1}l_k^j$ and $\lambda_{j,k}=C2^jV\left(B_k^j\right)$. Then $f=\sum_{j,k}\lambda_{j,k}a_k^j$, with a_k^j being $(1,q^*)$ homogeneous Hardy-Sobolev atoms and

$$\begin{aligned}
\sum_{j,k}|\lambda_{j,k}| &= C\sum_{j,k}2^jV\left(B_k^j\right)\\
&\leq C\sum_{j,k}2^jV(\underline{B_k^j})\\
&\leq C\sum_j 2^jV\left(\{x:\mathcal{M}_q(Nf)(x)>2^j\}\right)\\
&\leq C\sum_{x\in\Gamma}\mathcal{M}_q(Nf)(x)m(x)\\
&\leq C_q\left\|Nf\right\|_{L^1(\Gamma)}\sim\left\|f\right\|_{\dot{S}^{1,1}(\Gamma)},
\end{aligned}$$

where we used the fact that the $\underline{B_k^j}$ are pairwise disjoint. □

Proof of Lemma 5.7: let $x\in\underline{B_k^j}$ and $y\in B_l^{j+1}$ such that $x\sim y$. Denote by x_k^j (resp. x_l^{j+1}) the center of B_k^j (resp. B_l^{j+1}). Then

$$d(x_k^j,x_l^{j+1})\leq d(x_k^j,x)+d(x,y)+d(y,x_l^{j+1})\leq r_k^j+r_l^{j+1}+1.$$

Thus, since $F^j\subset F^{j+1}$,

$$\begin{aligned}
r_l^{j+1} &= \tfrac{1}{2}d\left(x_l^{j+1},F^{j+1}\right)\\
&\leq \tfrac{1}{2}d\left(x_l^{j+1},x_k^j\right)+\tfrac{1}{2}d(x_k^j,F^{j+1})\\
&\leq \tfrac{1}{2}\left(r_k^j+r_l^{j+1}+1\right)+\tfrac{1}{2}d(x_k^j,F^j),
\end{aligned}$$

from which we deduce

$$r_l^{j+1}\leq r_k^j+1+d(x_k^j,F^j)=r_k^j+1+2r_k^j\leq 4r_k^j,$$

as claimed. The proof of 2 is classical. □.

5.4 Comparison between different atomic spaces

In the present section, we show that $\dot{H}S^1_{t,ato}(\Gamma)=\dot{H}S^1_{t',ato}(\Gamma)$ for all $t,t'\in(1,+\infty]$, following ideas from [BB10]. We will need:

Lemma 5.8 *Assume that* Γ *satisfies* (D).

1. *Let*

$$\mathcal{M}_cf(x):=\sup_{r>0}\frac{1}{V(x,r)}\sum_{B(x,r)}|f(y)|m(y)$$

be the centered maximal function of f. Observe that if $x \in B(y,r)$ then $B(y,r) \subset B(x,2r)$. It follows that

$$\mathcal{M}_c f \leq \mathcal{M}_{HL} f \leq C\mathcal{M}_c f$$

where C only depends on the constant of the doubling property.

2. *Let f be an L^1 function supported in $B_0 = B(x_0, r_0)$. Then there is C_1 depending on the doubling constant such that*

$$\Omega_\alpha := \{x \in \Gamma : \mathcal{M}_{HL}(f)(x) > \alpha\} \subset B(x_0, 2r_0)$$

whenever $\alpha > \frac{C_1}{V(B_0)} \sum\limits_{x \in B_0} |f(x)| m(x)$

Proof: it is obvious that $\mathcal{M}_c f \leq \mathcal{M}_{HL} f$ everywhere on Γ. Moreover, let $x \in \Gamma$ et $B = B(x_0, r) \ni x$ be a ball. Then $B \subset B(x, 2r) \subset B(x_0, 3r)$, so that

$$\frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y) \leq \frac{1}{V(x,2r)} \frac{V(x,2r)}{V(B)} \sum_{y \in B(x,2r)} |f(y)|\, m(y) \leq C\mathcal{M}_c f(x),$$

and the result follows by taking the supremum over all balls B containing x.
For the second assertion, assume that $x \notin B(x_0, 2r_0)$ and let $B = B(x,r) \ni x$ be a ball centered at x. Then

$$\frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y) = \frac{1}{V(B)} \sum_{y \in B \cap B(x_0, r_0)} |f(y)|\, m(y).$$

If $B \cap B(x_0, r_0) = \emptyset$, then this quantity is 0. Otherwise, $2r_0 < d(x, x_0) \leq r + r_0$, so that $r_0 < r$ and $B_0 \subset B(x, 2r)$. It follows that

$$\frac{1}{V(B)} \sum_{y \in B} |f(y)|\, m(y) \leq \frac{1}{V(B_0)} \frac{V(B_0)}{V(B)} \sum_{y \in B_0} |f(y)|\, m(y) \leq \frac{C}{V(B_0)} \sum_{y \in B_0} |f(y)|\, m(y),$$

which yields the conclusion by part 1., provided that C_1 is big enough. □

Let us now prove:

Proposition 5.9 *Let Γ satisfying (D) and the Poincaré inquality (P_1). Then $HS^1_{t,ato} \subset HS^1_{\infty,ato}$ for every $t > 1$ and therefore $HS^1_{t_1,ato} = HS^1_{t_2,ato}$ for every $1 < t_1, t_2 \leq +\infty$.*

Proof: let $t > 1$. It is enough to prove that there exists $C > 0$ such that, for every $(1,t)-$atom a, a belongs to $\dot{H}S^1_{\infty,ato}(\Gamma)$ with

$$\|a\|_{\dot{H}S^1_{\infty,ato}(\Gamma)} \leq C.$$

In the sequel, set $\mathcal{M}^1_{HL} := \mathcal{M}_{HL}$ and $\mathcal{M}^{n+1}_{HL} = \mathcal{M}^n_{HL} \circ \mathcal{M}_{HL}$ for all $n \in \mathbb{N}$. Let a be $(1,t)-$atom supported in a ball B_0. Set $b = V(B_0)a$.
We claim that there exist $K, \alpha, C, N > 0$ only depending on t and the geometric constants with the following property: for all $l \in \mathbb{N}^*$, there exists a collection of balls $(B_{j_l})_{j_l \in \mathbb{N}^l}$ such that for every $n \geq 1$

$$b = CN \sum_{l=1}^{n-1} (K\alpha)^{l+1} \sum_{j_l \in \mathbb{N}^l} V(B_{j_l}) a_{j_l} + \sum_{j_n \in \mathbb{N}^n} h_{j_n} \tag{5.58}$$

and, for all $n \in \mathbb{N}^*$,

$$a_{j_l} \text{ is an } (1,\infty)\text{-atom supported in } B_{j_l}, 1 \leq l \leq n-1, \tag{5.59}$$

$$\bigcup_{j_n \in \mathbb{N}^n} B_{j_n} \subset \Omega_l := \left\{ x \in \Gamma; \mathcal{M}_{HL}^{l+1}(|\nabla b|)(x) > K\frac{\alpha^l}{2} \right\}, \tag{5.60}$$

$$\sum_{j_l} \mathbf{1}_{B_{j_l}} \leq N^l, \tag{5.61}$$

$$\text{supp } h_{j_l} \subset B_{j_l}, \sum_{x \in B_{j_l}} h_{j_l}(x)m(x) = 0, \tag{5.62}$$

$$|\nabla h_{j_l}(x)| \leq C\left((\alpha K)^l \chi_{j_l} + \mathcal{M}_{HL}^l(|\nabla b|)\right)(x) \text{ for all } x \in \Gamma, \tag{5.63}$$

$$\frac{1}{V(B_{j_l})} \left\| \nabla h_{j_l} \right\|_{L^1(\Gamma)} \leq C(K\alpha)^l, \tag{5.64}$$

where χ_{j_n} stands for the characteristic function of B_{j_n}.

Let us assume that this construction is done. We claim that

$$a = \sum_{l=1}^{\infty} CN(K\alpha)^{l+1} \sum_{j_l \in \mathbb{N}^l} \frac{V(B_{j_l})}{V(B_0)} a_{j_l}, \tag{5.65}$$

where the series converges in $\dot{W}^{1,1}(\Gamma)$ and

$$\frac{N}{V(B_0)} \sum_{l=1}^{\infty} (K\alpha)^{l+1} \sum_{j_l \in \mathbb{N}^l} V(B_{j_l}) \leq C \tag{5.66}$$

where C is independent of a.

Let us first check (5.66). Indeed, it follows from (5.59), (5.61) and the $L^t(\Gamma)$-boundedness of $\mathcal{M}_{HL}$ that

$$\sum_{j_l} V(B_{j_l}) \leq CN^l m\left(\bigcup_{j_l} B_{j_l}\right) \leq CN^l m(\Omega_l) \leq CN^l \left(\frac{2}{K\alpha^l}\right)^t \|\nabla b\|_{L^t(\Gamma)}^t.$$

As a consequence,

$$\begin{aligned} \sum_{l=0}^{\infty} (K\alpha)^l \sum_{j_l \in \mathbb{N}^l} V(B_{j_l}) &\leq C2^t \sum_{n=0}^{\infty} (K\alpha)^l N^l (K\alpha^l)^{-t} \|\nabla b\|_{L^t(\Gamma)}^t \\ &\leq C2^t K^{-t} \sum_{l=0}^{\infty} (NK\alpha^{(1-t)})^l \|\nabla b\|_{L^t(\Gamma)}^t \end{aligned}$$

and, since $\|\nabla b\|_{L^t(\Gamma)}^t \leq CV(B_0)$, we obtain (5.66) with C only depending on t, K, α and N, provided that α is chosen such that $\frac{NK}{\alpha^{t-1}} < 1$.

We now focus on (5.65). By (5.64), one has

$$\begin{aligned}\frac{1}{V(B_0)}\left\|\sum_{j_n\in\mathbb{N}^n} h_{j_n}\right\|_{\dot{W}^{1,1}(\Gamma)} &\leq \frac{1}{V(B_0)}\sum_{j_n\in\mathbb{N}^n}\left\|\nabla h_{j_n}\right\|_{L^1(\Gamma)}\\ &\leq C(K\alpha)^n\sum_{j_n\in\mathbb{N}^n}\frac{V(B_{j_n})}{V(B_0)}\end{aligned}$$

and, by (5.66), this quantity converges to 0 when $n\to+\infty$, which yields (5.65).

Let us now turn the the construction, which will be done by induction on l, starting with $l=1$. Set

$$\widetilde{\Omega}_1=\{x\in\Gamma:\mathcal{M}_{HL}(\nabla b)(x)>K\alpha\},$$

where K,α will be chosen such that $K\alpha>C_1$ and C_1 is given by Lemma 5.8. Hence, $\widetilde{\Omega}_1\subset 2B_0$. Moreover,

$$m\left(\widetilde{\Omega}_1\right)\leq\frac{1}{(K\alpha)^t}\left\|\mathcal{M}_{HL}(\nabla b)\right\|^t_{L^t(\Gamma)}\leq\frac{C}{(K\alpha)^t}\left\|(\nabla b)\right\|^t_{L^t(\Gamma)}<+\infty.$$

If $\widetilde{\Omega_1}=\emptyset$, then $\frac{b}{NCK\alpha V(B_0)}$ is a $(1,\infty)$ atom and we are done. Assume now that $\widetilde{\Omega_1}\neq\emptyset$ and define the balls B_i and the functions χ_i as in the proof of Proposition 4.1. Set also

$$h_i:=(b-c_i)\chi_i,$$

where

$$c_i:=\frac{1}{\sum_{x\in B_i}\chi_i(x)m(x)}\sum_{x\in B_i}b(x)\chi_i(x)m(x).$$

Clearly, supp $h_i\subset B_i$. Moreover,

$$\sum_{x\in B_i}h_i(x)m(x)=0. \tag{5.67}$$

We now claim:

$$\left\|\nabla h_i\right\|_{L^1(\Gamma)}\leq C\alpha V(B_i). \tag{5.68}$$

Indeed, arguing as in the proof of Proposition 5.6, one has, for all $x\sim y\in\Gamma$,

$$\begin{aligned}b_i(y)-b_i(x) &= ((b(y)-b(x))\chi_i(y)+(b(x)-c_i)(\chi_i(y)-\chi_i(x))\\ &= A(x,y)+B(x,y).\end{aligned}$$

On the one hand, using the support condition on χ_i,

$$\sum_{x\sim y}\left|A(x,y)\right|m(x)\leq\sum_{x\in 2B_i}\left|\nabla b(x)\right|m(x)\leq CV(B_i)K\alpha. \tag{5.69}$$

On the other hand,

$$\sum_{x\sim y}\left|B(x,y)\right|m(x)\leq\frac{C}{r_i}\sum_{x\in 2B_i}\left|b(x)-c_i\right|m(x).$$

But

$$\|b-c_i\|_{L^1(2B_i)} \leq \|b-b_{B_i}\|_{L^1(2B_i)} + CV(B_i)\left|b_{B_i}-c_i\right|,$$

and, arguing as in the proof of Proposition 5.6 and using (P_1), one obtains

$$\sum_{x\sim y} |B(x,y)|\, m(x) \leq \sum_{x\in 2B_i} |\nabla b(x)|\, m(x) \leq CV(B_i)K\alpha. \tag{5.70}$$

Thus, (5.69) and (5.70) yield (5.68).

Define now the functions g (denoted by g_0 in the sequel) and h as in the proof of Proposition 4.1, so that

$$b=\sum_j h_j + g_0. \tag{5.71}$$

Observe that the series in (5.71) converges in $\dot{W}^{1,1}(\Gamma)$. Indeed, by (5.68),

$$\begin{aligned}
\sum_j \|\nabla h_j\|_{L^1(\Gamma)} &= \sum_j \|\nabla h_j\|_{L^1(2B_j)} \\
&\leq C\sum_j V(B_j)^{1-1/t}\, \|\nabla h_j\|_{L^t(2B_j)} \\
&\leq CK\alpha \sum_j V(B_j) \\
&\leq C(K\alpha)^{1-t}\, \|\nabla b\|_{L^t(\Gamma)}^t \\
&\leq C(K\alpha)^{1-t} V(B_0).
\end{aligned}$$

Moreover, since $\sum b(x)m(x)=0$ and $\sum h_j(x)m(x)=0$ for all j, one also has $\sum g_0(x)m(x)=0$. Arguing as in the proof of Proposition 4.1, one establishes that

$$\|\nabla g_0\|_{L^\infty(\Gamma)} \leq CK\alpha.$$

It follows that $a_0=\frac{g_0}{NCK\alpha V(B_0)}$ is a $(1,\infty)$-atom, and (5.71) yields

$$b=NCK\alpha V(B_0)a_0+\sum_{j\in\mathbb{N}} h_j$$

Thus, properties (5.59), (5.60), (5.61) and (5.62) hold. Property (5.64) has already been checked in (5.68). Moreover,

$$\begin{aligned}
|\nabla h_j(x)| &\leq |b(x)-c_j||\nabla\chi_j(x)| + (\max_{y\sim x}\chi_j(y))|\nabla b(x)| \\
&= I+II.
\end{aligned}$$

We estimate I as follows:

$$I\leq \frac{C}{r_j}|b(x)-c_j|.$$

But, following the proof of Theorem 0.1 in [BB10], if $l_j \in \mathbb{Z}$ is such that $2^{l_j} \leq r_j < 2^{l_j+1}$, one has, using (P_1),

$$\begin{aligned}
|b(x) - c_j| &\leq \sum_{k=-(l_j+1)}^{-1} \left| b_{B(x,2^k r_j)} - b_{B(x,2^{k+1} r_j)} \right| + \left| b_{B(x,r_j)} - c_j \right| \\
&\leq \sum_{k=-(l_j+1)}^{-1} \frac{1}{V(x,2^k r_j)} \sum_{z \in B(x,2^k r_j)} \left| b(z) - b_{B(x,2^{k+1} r_j)} \right| m(z) + \left| b_{B(x,r_j)} - b_{2B_j} \right| \\
&+ \left| \frac{1}{\sum_{z \in B_j} \chi_j(z) m(z)} \sum_{z \in B_j} \left(b(z) - \frac{1}{V(2B_j)} \sum_{w \in 2B_j} b(w) m(w) \right) \chi_j(z) m(z) \right| \\
&\leq C \sum_{k=-(l_j+1)}^{-1} \frac{1}{V(x,2^{k+1} r_j)} \sum_{z \in B(x,2^{k+1} r_j)} \left| b(z) - b_{B(x,2^{k+1} r_j)} \right| m(z) + \left| b_{B(x,r_j)} - b_{2B_j} \right| \\
&+ \left| \frac{1}{\sum_{z \in B_j} \chi_j(z) m(z)} \sum_{z \in B_j} \left(b(z) - \frac{1}{V(2B_j)} \sum_{w \in 2B_j} b(w) m(w) \right) \chi_j(z) m(z) \right| \\
&\leq C \sum_{k=-(l_j+1)}^{-1} 2^{k+1} r_j \mathcal{M}_{HL}(|\nabla b|)(x) + \frac{1}{V(2B_j)} \sum_{z \in 2B_j} \left| b(z) - b_{2B_j} \right| m(z) \\
&+ \frac{1}{\sum_{z \in B_j} \chi_j(z) m(z)} \sum_{z \in 2B_j} \left| b(z) - \frac{1}{V(2B_j)} \sum_{w \in 2B_j} b(w) m(w) \right| \left| \chi_j(z) \right| m(z) \\
&\leq C r_j \left(\mathcal{M}_{HL}(|\nabla b|)(x) + K\alpha \right).
\end{aligned}$$

Moreover, $II \leq |\nabla b(x)| \leq \mathcal{M}_{HL}(|\nabla b|)(x)$. Finally, (5.63) is satisfied. The construction for $l = 1$ is therefore complete.

Assuming now that the construction is done for l, the construction for $l+1$ is performed by arguments analogous to the previous one (see also the proof of Theorem 0.1 in [BB10]). This ends the proof of Proposition 5.9. □

5.5 Interpolation between Hardy-Sobolev and Sobolev spaces

To establish Theorem 2.14, observe that, by Theorems 2.9 and 2.12, $f \in \dot{S}^{1,1}(\Gamma)$ (resp. $f \in \dot{W}^{1,p}(\Gamma)$ if $p > 1$) if and only if $\mathcal{M}^+ f \in L^1(\Gamma)$ (resp. $\mathcal{M}^+ f \in L^p(\Gamma)$. Therefore, Theorem 2.14 follows from the classical linearization method of maximal operators (see [SW71], Chapter 5).

6 Boundedness of Riesz transforms

6.1 The boundedness of Riesz transforms on Hardy-Sobolev spaces

This section is devoted to the proof of Theorem 2.15. We first establish:

Proposition 6.1 *There exists $C>0$ such that, for all atom $a\in H^1(\Gamma)$, $(I-P)^{-1/2}a\in \dot{S}^{1,1}(\Gamma)$ and*

$$\left\|(I-P)^{-1/2}a\right\|_{\dot{S}^{1,1}(\Gamma)}\leq C. \tag{6.72}$$

The proof relies on some estimates for the iterates of p, taken from [Rus00, Rus01]. Define

$$p_0(x,y):=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{if } x\neq y,\end{cases}$$

and, for all $k\in\mathbb{N}$ and all $x,y\in\Gamma$,

$$p_{k+1}(x,y)=\sum_{z\in\Gamma}p(x,z)p_k(z,y).$$

By (2.5), one has

$$p_k(x,y)m(x)=p_k(y,x)m(y)$$

for all $k\in\mathbb{N}$ and all $x,y\in\Gamma$.
Let $y_0\in\Gamma$. For all $k\in\mathbb{N}$ and all $x\in\Gamma$, define

$$q_k(x,y):=\frac{p_k(y,x)-p_k(y_0,x)}{m(x)}.$$

Recall the following bounds on p_k and q_k ([Rus00], Lemmata 2 and 4 and [Rus01], Lemmata 28 and 29):

Lemma 6.2 *There exist $C,\alpha>0$ such that, for all $y\in\Gamma$,*

1. $$\sum_{x\in\Gamma}\left|\nabla_x p_k(x,y)\right|^2\exp\left(\alpha\frac{d^2(x,y)}{k}\right)m(x)\leq\frac{C}{V(y,\sqrt{k})}m^2(y),$$

2. $$\sum_{x\in\Gamma}\left|\nabla_x p_k(x,y)\right|^2\exp\left(\alpha\frac{d^2(x,y)}{k}\right)m(x)\leq\frac{C}{kV(y,\sqrt{k})}m^2(y).$$

Lemma 6.3 *There exist $C,h,\alpha>0$ such that, for all $y_0,y\in\Gamma$ and all $k\geq 1$ such that $d(y,y_0)\leq\sqrt{k}$,*

1. $$\sum_{x\in\Gamma}\left|q_k(x,y)\right|^2\exp\left(\alpha\frac{d^2(x,y)}{k}\right)m(x)\leq\frac{C}{V(y,\sqrt{k})}\left(\frac{d(y,y_0)}{\sqrt{k}}\right)^h,$$

2. $$\sum_{x\in\Gamma}\left|\nabla_x q_k(x,y)\right|^2\exp\left(\alpha\frac{d^2(x,y)}{k}\right)m(x)\leq\frac{C}{kV(y,\sqrt{k})}\left(\frac{d(y,y_0)}{\sqrt{k}}\right)^h.$$

Proof of Proposition 6.1: let a be an atom supported in $B = B(y_0, r)$. Pick up a sequence of functions $(\chi_j)_{j\geq 0}$ such that

$$\text{supp } \chi_0 \in 4B, \ \text{supp } \chi_j \subset 2^{j+2}B \setminus 2^{j-1}B, \ \|d\chi_j\|_\infty \leq \frac{C}{2^j r}$$

and

$$\sum_{j\geq 0} \chi_j = 1 \text{ on } \Gamma.$$

For all $j \geq 0$, all $x \in \Gamma$ and all $y \sim x$, one has

$$\begin{aligned}\chi_j(y)(I-P)^{-1/2}a(y) - \chi_j(x)(I-P)^{-1/2}a(x) &= \chi_j(y)\left((I-P)^{-1/2}a(y) - (I-P)^{-1/2}a(x)\right) \\ &+ (I-P)^{-1/2}a(x)\left(\chi_j(y) - \chi_j(x)\right).\end{aligned}$$

It follows that, if $\nabla\left(\chi_j(I-P)^{-1/2}a\right)(x) \neq 0$, then either $\chi_j(x) \neq 0$, or there exists $y \sim x$ such that $\chi_j(y) \neq 0$. As a consequence, supp $\nabla\left(\chi_j(I-P)^{-1/2}a\right) \subset C_j(B) := 2^{j+3}B \setminus 2^{j-2}B$ if $j \geq 3$ and supp $\nabla\left(\chi_j(I-P)^{-1/2}a\right) \subset C_j(B) := 2^{j+3}B$ if $j \leq 2$. Decompose $(I-P)^{-1/2}a$ as

$$\begin{aligned}(I-P)^{-1/2}a &= \sum_{j\geq 0} \chi_j(I-P)^{-1/2}a \\ &= \sum_{j\geq 0} V^{1/2}(2^{j+3}B)\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)} \frac{\chi_j(I-P)^{-1/2}a}{V^{1/2}(2^{j+3}B)\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)}} \\ &:= \sum_{j\geq 0} V^{1/2}(2^{j+3}B)\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)} b_j.\end{aligned}$$

We first check that, for all $j \geq 0$, up to a constant only depending on the constants of the graph Γ, b_j is an atom in $\dot{H}S^1_{2,ato}(\Gamma)$ if, in Definition 2.11, condition 3 is replaced by condition $3'$ in Remark 2.13. Indeed, since (D) and (P_1) hold, there exists $C > 0$ such that, for all balls B of radius r and all functions $f \in W^{1,2}_0(B)$,

$$\|f\|_{L^2(B)} \leq Cr\,\|\nabla f\|_{L^2(B)} \tag{6.73}$$

(see [BR09], inequality (8.2)). Then, for all j, since χ_j is supported in $2^{j+2}B$, (6.73) yields

$$\left\|\chi_j(I-P)^{-1/2}a\right\|_{L^2(2^{j+2}B)} \leq C2^{j+2}r\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)},$$

which shows that

$$\|b_j\|_{L^2(2^{j+2}B)} \leq C2^{j+2}rV^{-1/2}(2^{j+2}B),$$

as claimed.

The estimate (6.72) will therefore be a consequence of

$$\sum_{j\geq 0} V^{1/2}(2^{j+3}B)\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)} \leq C. \tag{6.74}$$

Write

$$\begin{aligned}\left\|\nabla\left(\chi_j(I-P)^{-1/2}a\right)\right\|_{L^2(\Gamma)} &\leq \left\|\nabla(I-P)^{-1/2}a\right\|_{L^2(C_j(B))}+\|\nabla\chi_j\|_\infty\left\|(I-P)^{-1/2}a\right\|_{L^2(C_j(B))}\\ &:= S_j+T_j.\end{aligned}$$

Let us first focus on T_j. As in [BR09], we use the expansion

$$\begin{aligned}(I-P)^{-1/2}a &= \sum_{k=0}^{+\infty} a_k P^k a\\ &= \sum_{k=0}^{r^2} a_k P^k a+\sum_{k=r^2+1}^{+\infty} a_k P^k a\\ &:= f_1+f_2,\end{aligned}$$

where the a_k's are defined by

$$(1-x)^{-1/2}=\sum_{k=0}^{+\infty} a_k x^k$$

for all $x\in(-1,1)$. Recall that, when $k\to+\infty$,

$$a_k\sim\frac{1}{\sqrt{k\pi}}. \tag{6.75}$$

For f_1,

$$\|f_1\|_{L^2(C_j(B))}\leq\sum_{k=0}^{r^2} a_k\left\|P^k a\right\|_{L^2(C_j(B))}. \tag{6.76}$$

For $k=0$, $P^k a=a$ so that

$$\left\|P^k a\right\|_{L^2(\Gamma)}\leq V(B)^{-1/2}.$$

Let $h\in L^2(C_j(B))$ with $\|h\|_{L^2}\leq 1$. For all $1\leq k\leq r^2$, Lemma 6.2 yields

$$\begin{aligned}\left|\sum_{x\in C_j(B)} P^k a(x)h(x)m(x)\right| &\leq \sum_{x\in C_j(B)}|h(x)|\left(\sum_{y\in\Gamma}p_k(x,y)\,|a(y)|\right)m(x)\\ &= \sum_{y\in\Gamma}|a(y)|\left(\sum_{x\in C_j(B)}p_k(x,y)\exp\left(\frac{\alpha d^2(x,y)}{2k}\right)\exp\left(-\frac{\alpha d^2(x,y)}{2k}\right)\right.\\ &\qquad \left.|h(x)|\,m(x)\right)\\ &\leq e^{-c\frac{2^{2j}r^2}{k}}\sum_{y\in\Gamma}|a(y)|\left(\sum_{x\in 2^{j+3}B}|p_k(x,y)|^2\exp\left(\frac{\alpha d^2(x,y)}{k}\right)m(x)\right)^{1/2}\\ &\qquad \|h\|_{L^2(C_j(B))}\\ &\leq Ce^{-c\frac{2^{2j}r^2}{k}}\sum_{y\in B}\frac{|a(y)|}{V^{1/2}(y,\sqrt{k})}m(y).\end{aligned} \tag{6.77}$$

But, for all $y \in B$, (D) shows that

$$\begin{aligned} \frac{1}{V(y,\sqrt{k})} &= \frac{1}{V(y_0,\sqrt{k})}\frac{V(y_0,\sqrt{k})}{V(y,\sqrt{k})} \\ &\leq \frac{1}{V(y_0,\sqrt{k})}\frac{V(y,\sqrt{k}+r)}{V(y,\sqrt{k})} \\ &\leq \frac{1}{V(y_0,\sqrt{k})}\left(1+\frac{r}{\sqrt{k}}\right)^D \\ &\leq \frac{1}{V(2^{j+3}B)}\frac{V(2^{j+3}B)}{V(y_0,\sqrt{k})}\left(1+\frac{r}{\sqrt{k}}\right)^D \\ &\leq \frac{1}{V(2^{j+3}B)}\left(1+\frac{2^{j+3}r}{\sqrt{k}}\right)^{2D}. \end{aligned} \tag{6.78}$$

Therefore, it follows from (6.77) and the fact that $\|a\|_1 \leq 1$ that

$$\left\|P^k a\right\|_{L^2(C_j(B))} \leq \frac{C}{V^{1/2}(2^{j+3}B)}\exp\left(-c'\frac{2^{2j}r^2}{k}\right).$$

Since, when $j \geq 3$ and $k=0$, $P^k a = a$ and $C_j(B)$ are disjoint, one obtains

$$\|f_1\|_{L^2(C_j(B))} \leq \frac{C}{V^{1/2}(2^{j+3}B)}\left(\sum_{k=1}^{r^2}\frac{1}{\sqrt{k}}\exp\left(c'\frac{2^{2j}r^2}{k}\right)+c_j\right), \tag{6.79}$$

with $c_j = 1$ if $j \leq 2$ and $c_j = 0$ if $j \geq 3$.
For f_2,

$$\|f_2\|_{L^2(C_j(B))} \leq \sum_{k=r^2+1}^{\infty} a_k \left\|\sum_{y\in\Gamma} q_k(\cdot,y)a(y)m(y)\right\|_{L^2(C_j(B))}.$$

Pick up a function $h \in L^2(C_j(B))$ with $\|h\|_{L^2} \leq 1$ again. For all $k \geq r^2+1$, Lemma 6.3 yields

$$\begin{aligned} \left|\sum_{x\in C_j(B)} P^k a(x)h(x)m(x)\right| &\leq \sum_{x\in C_j(B)} |h(x)|\left(\sum_{y\in\Gamma} q_k(x,y)\,|a(y)|\,m(y)\right)m(x) \\ &= \sum_{y\in\Gamma}|a(y)|\Bigg(\sum_{x\in C_j(B)} q_k(x,y)\exp\left(\frac{\alpha d^2(x,y)}{2k}\right)\exp\left(-\frac{\alpha d^2(x,y)}{2k}\right) \\ &\quad |h(x)|\,m(x)\Bigg)m(y) \\ &\leq e^{-c\frac{2^{2j}r^2}{k}}\sum_{y\in\Gamma}|a(y)|\left(\sum_{x\in C_j(B)}|q_k(x,y)|^2\exp\left(\frac{\alpha d^2(x,y)}{k}\right)m(x)\right)^{1/2} \\ &\times \|h\|_{L^2(C_j(B))}\,m(y) \\ &\leq Ce^{-c\frac{2^{2j}r^2}{k}}\left(\frac{r}{\sqrt{k}}\right)^{h/2}\sum_{y\in B}\frac{|a(y)|}{V^{1/2}(y,\sqrt{k})}m(y) \\ &\leq \frac{C}{V^{1/2}(y_0,\sqrt{k})}e^{-c\frac{2^{2j}r^2}{k}}\left(\frac{r}{\sqrt{k}}\right)^{h/2}. \end{aligned} \tag{6.80}$$

Arguing as before and using (2.21), one therefore obtains

$$\begin{aligned}
\|f_2\|_{L^2(C_j(B))} &\leq \sum_{k=r^2+1}^{+\infty} \frac{C}{\sqrt{k}V^{1/2}(y_0,\sqrt{k})} e^{-c\frac{2^{2j}r^2}{k}} \left(\frac{r}{\sqrt{k}}\right)^{h/2} \\
&= \frac{C}{V^{1/2}(2^{j+3}B)} \sum_{k=r^2+1}^{+\infty} \frac{1}{\sqrt{k}} \frac{V^{1/2}(2^{j+3}B)}{V^{1/2}(y_0,\sqrt{k})} e^{-c\frac{2^{2j}r^2}{k}} \left(\frac{r}{\sqrt{k}}\right)^{h/2} \\
&\leq \frac{C}{V^{1/2}(2^{j+3}B)} \sum_{k=r^2+1}^{+\infty} \frac{1}{\sqrt{k}} f\left(\frac{2^{j+3}r}{\sqrt{k}}\right) e^{-c\frac{2^{2j}r^2}{k}} \left(\frac{r}{\sqrt{k}}\right)^{h/2},
\end{aligned} \tag{6.81}$$

where

$$f(u) = \begin{cases} u^{D/2} & \text{if } u > 1, \\ u^{d/2} & \text{if } u \leq 1. \end{cases}$$

Gathering (6.79) and (6.81), one therefore obtains

$$T_j \leq \frac{C}{2^j r V^{1/2}(2^{j+3}B)} \left(c_j + \sum_{k=1}^{r^2} \frac{1}{\sqrt{k}} e^{-c\frac{2^{2j}r^2}{k}} + \sum_{k=r^2+1}^{+\infty} \frac{1}{\sqrt{k}} e^{-c\frac{2^{2j}r^2}{k}} f\left(\frac{2^{j+3}r}{\sqrt{k}}\right) \left(\frac{r}{\sqrt{k}}\right)^{h/2} \right).$$

Thus,

$$\begin{aligned}
V^{1/2}(2^{j+3}B)T_j &\leq \frac{C}{2^j r} \left(c_j + \sum_{k=1}^{r^2} \frac{1}{\sqrt{k}} e^{-c\frac{2^{2j}r^2}{k}} + \sum_{k=r^2+1}^{+\infty} \frac{1}{\sqrt{k}} e^{-c\frac{2^{2j}r^2}{k}} f\left(\frac{2^{j+3}r}{\sqrt{k}}\right) \left(\frac{r}{\sqrt{k}}\right)^{h/2} \right) \\
&\leq \frac{C}{2^j r} \int_0^{2r^2} e^{-c\frac{2^{2j}r^2}{t}} \frac{dt}{\sqrt{t}} + \int_{r^2}^{+\infty} e^{-c\frac{2^{2j}r^2}{t}} f\left(\frac{2^{j+3}r}{\sqrt{t}}\right) \left(\frac{r}{\sqrt{t}}\right)^{h/2} \frac{dt}{t} \\
&= C \int_{2^{2j-1}}^{+\infty} e^{-cu} \frac{du}{u^{3/2}} + C \int_0^{2^{2j}} e^{-cu} f(8\sqrt{u}) \left(\frac{\sqrt{u}}{2^j}\right)^{h/2} \frac{du}{u^{3/2}} \\
&\leq C 2^{-jh/2}.
\end{aligned}$$

Note that we used the fact that $d \geq 1$ in the last inequality (this is the only place where this assumption is used). Finally,

$$\sum_{j=0}^{+\infty} V^{1/2}(2^{j+3}B)T_j \leq C. \tag{6.82}$$

Let us now focus on S_j. For $j \leq 2$, the L^2-boundedness of $\nabla(I-P)^{-1/2}$ yields

$$\left\|\nabla(I-P)^{-1/2}a\right\|_{L^2(C_j(B))} \leq \left\|\nabla(I-P)^{-1/2}a\right\|_{L^2(\Gamma)} \leq CV(B)^{-1/2}.$$

Take now $j \geq 3$. As before, one has

$$\begin{aligned}
\nabla(I-P)^{-1/2}a &\leq \sum_{k=0}^{+\infty} a_k \nabla P^k a \\
&= \sum_{k=0}^{r^2} a_k \nabla P^k a + \sum_{k=r^2+1}^{+\infty} a_k \nabla P^k a \\
&:= g_1 + g_2.
\end{aligned}$$

We estimate the L^2-norms of g_1 and g_2. For g_1,

$$\|g_1\|_{L^2(C_j(B))} \leq \sum_{k=0}^{r^2} a_k \left\|\nabla P^k a\right\|_{L^2(C_j(B))}. \tag{6.83}$$

Notice that, when $k=0$, $\nabla P^k a = \nabla a$ is supported in $2B$, which is disjoint from $C_j(B)$ since $j\geq 3$. Let $h\in L^2(C_j(B))$ with $\|h\|_{L^2}\leq 1$. For all $1\leq k\leq r^2$, Lemma 6.2 yields

$$\begin{aligned}
\left|\sum_{x\in C_j(B)} \nabla P^k a(x) g(x) m(x)\right| &\leq \sum_{x\in C_j(B)} |h(x)| \left(\sum_{y\in\Gamma} \nabla_x p_k(x,y) \left|a(y)\right|\right) m(x)\\
&= \sum_{y\in\Gamma} |a(y)| \left(\sum_{x\in C_j(B)} \nabla_x p_k(x,y) \exp\left(\frac{\alpha d^2(x,y)}{2k}\right) \exp\left(-\frac{\alpha d^2(x,y)}{2k}\right)\right.\\
&\quad \left. |h(x)|\, m(x)\right)\\
&\leq e^{-c\frac{2^{2j}r^2}{k}} \sum_{y\in\Gamma} |a(y)| \left(\sum_{x\in C_j(B)} \left|\nabla_x p_k(x,y)\right|^2 \exp\left(\frac{\alpha d^2(x,y)}{k}\right) m(x)\right)^{1/2}\\
&\quad \|h\|_{L^2(C_j(B))}\\
&\leq \frac{C}{\sqrt{k}} e^{-c\frac{2^{2j}r^2}{k}} \sum_{y\in B} \frac{|a(y)|}{V^{1/2}(y,\sqrt{k})} m(y).
\end{aligned} \tag{6.84}$$

Thus, it follows from (6.78), (6.84) and the fact that $\|a\|_1\leq 1$ that

$$\left\|\nabla P^k a\right\|_{L^2(C_j(B))} \leq \frac{C}{\sqrt{k} V^{1/2}(2^{j+3}B)} \exp\left(-c'\frac{2^{2j}r^2}{k}\right).$$

As a consequence of (6.75) and (6.83), one therefore has

$$\|g_1\|_{L^2(C_j(B))} \leq \frac{C}{V^{1/2}(2^{j+3}B)} \left(c_j + \sum_{k=1}^{r^2} \frac{1}{k} \exp\left(c'\frac{2^{2j}r^2}{k}\right)\right), \tag{6.85}$$

where, again, $c_j=1$ if $j\leq 2$ and $c_j=0$ if $j\geq 3$.
For g_2, observe that, for all $x\in\Gamma$, since $\sum_{y\in\Gamma} a(y)m(y)=0$,

$$\begin{aligned}
P^k a(x) &= \sum_{y\in\Gamma} \frac{p_k(x,y)}{m(y)} a(y) m(y)\\
&= \sum_{y\in\Gamma} \left(\frac{p_k(x,y)}{m(y)} - \frac{p_k(x,y_0)}{m(y_0)}\right) a(y) m(y)\\
&= \frac{1}{m(x)} \sum_{y\in\Gamma} \left(p_k(y,x) - p_k(y_0,x)\right) a(y) m(y)\\
&= \sum_{y\in\Gamma} q_k(x,y) a(y) m(y).
\end{aligned}$$

As a consequence,

$$\left\| g_2 \right\|_{L^2(C_j(B))} \leq \sum_{k=r^2+1}^{\infty} a_k \left\| \sum_{y\in\Gamma} \nabla_x q_k(\cdot,y) a(y) m(y) \right\|_{L^2(2^{j+3}B)}.$$

Pick up a function $h \in L^2(C_j(B))$ with $\left\| h \right\|_{L^2} \leq 1$ again. For all $k \geq r^2+1$, Lemma 6.3 yields

$$\begin{aligned}
\left| \sum_{x\in C_j(B)} \nabla P^k a(x) h(x) m(x) \right| &\leq \sum_{x\in C_j(B)} |h(x)| \left(\sum_{y\in\Gamma} \nabla_x q_k(x,y) \left| a(y) \right| m(y) \right) m(x) \\
&= \sum_{y\in\Gamma} |a(y)| \Bigg(\sum_{x\in C_j(B)} \nabla_x q_k(x,y) \exp\left(\frac{\alpha d^2(x,y)}{2k} \right) \exp\left(-\frac{\alpha d^2(x,y)}{2k} \right) \\
&\quad |h(x)| \, m(x) \Bigg) m(y) \\
&\leq e^{-c\frac{2^{2j} r^2}{k}} \sum_{y\in\Gamma} |a(y)| \left(\sum_{x\in C_j(B)} \left| \nabla_x q_k(x,y) \right|^2 \exp\left(\frac{\alpha d^2(x,y)}{k} \right) m(x) \right)^{1/2} \\
&\times \left\| h \right\|_{L^2(C_j(B))} m(y) \\
&\leq \frac{C}{\sqrt{k}} e^{-c\frac{2^{2j} r^2}{k}} \left(\frac{r}{\sqrt{k}} \right)^{h/2} \sum_{y\in B} \frac{|a(y)|}{V^{1/2}(y,\sqrt{k})} m(y) \\
&\leq \frac{C}{\sqrt{k} V^{1/2}(y_0,\sqrt{k})} e^{-c\frac{2^{2j} r^2}{k}} \left(\frac{r}{\sqrt{k}} \right)^{h/2}.
\end{aligned} \tag{6.86}$$

Using (6.75) again, as well as (D) and (2.21), one therefore obtains

$$\begin{aligned}
\left\| g_2 \right\|_{L^2(C_j(B))} &\leq \sum_{k=r^2+1}^{+\infty} \frac{C}{k V^{1/2}(y_0,\sqrt{k})} e^{-c\frac{2^{2j} r^2}{k}} \left(\frac{r}{\sqrt{k}} \right)^{h/2} \\
&= \frac{C}{V^{1/2}(2^{j+3}B)} \sum_{k=r^2+1}^{+\infty} \frac{1}{k} \frac{V^{1/2}(2^{j+3}B)}{V^{1/2}(y_0,\sqrt{k})} e^{-c\frac{2^{2j} r^2}{k}} \left(\frac{r}{\sqrt{k}} \right)^{h/2} \\
&\leq \frac{C}{V^{1/2}(2^{j+3}B)} \sum_{k=r^2+1}^{+\infty} \frac{1}{k} f\left(\frac{2^{j+3} r}{\sqrt{k}} \right) e^{-c\frac{2^{2j} r^2}{k}} \left(\frac{r}{\sqrt{k}} \right)^{h/2},
\end{aligned} \tag{6.87}$$

Gathering (6.85) and (6.87), one therefore obtains

$$S_j \leq \frac{C}{V^{1/2}(2^{j+3}B)} \left(c_j + \sum_{k=1}^{r^2} \frac{1}{k} e^{-c\frac{2^{2j} r^2}{k}} + \sum_{k=r^2+1}^{+\infty} \frac{1}{k} e^{-c\frac{2^{2j} r^2}{k}} f\left(\frac{2^{j+3} r}{\sqrt{k}} \right) \left(\frac{r}{\sqrt{k}} \right)^{h/2} \right).$$

Thus,

$$
\begin{aligned}
V^{1/2}(2^{j+3}B)S_j &\leq C\left(c_j+\sum_{k=1}^{r^2}\frac{1}{k}e^{-c\frac{2^{2j}r^2}{k}}+\sum_{k=r^2+1}^{+\infty}\frac{1}{k}e^{-c\frac{2^{2j}r^2}{k}}f\left(\frac{2^{j+3}r}{\sqrt{k}}\right)\left(\frac{r}{\sqrt{k}}\right)^{h/2}\right)\\
&\leq C\int_0^{2r^2}e^{-c\frac{2^{2j}r^2}{t}}\frac{dt}{t}+C\int_{r^2}^{+\infty}e^{-c\frac{2^{2j}r^2}{t}}f\left(\frac{2^{j+3}r}{\sqrt{t}}\right)\left(\frac{r}{\sqrt{k}}\right)^{h/2}\frac{dt}{t}\\
&= C\int_{2^{2j-1}}^{+\infty}e^{-cu}\frac{du}{u}+C\int_0^{2^{2j}}e^{-cu}f(8\sqrt{u})\left(\frac{\sqrt{u}}{2^j}\right)^{h/2}\frac{du}{u}\\
&\leq C2^{-jh/2},
\end{aligned}
$$

which proves that

$$\sum_{j=0}^{+\infty}V^{1/2}(2^{j+3}B)S_j\leq C. \tag{6.88}$$

Finally, (6.88) and (6.82) yield (6.74) and the proof of Proposition 6.1 is complete. □

Let us now derive Theorem 2.15 from Proposition 6.1. Take $f\in H^1(\Gamma)$ and decompose

$$f=\sum_{j=0}^{+\infty}\lambda_j a_j$$

with $\sum_{j=0}^{+\infty}|\lambda_j|\leq 2\left\|f\right\|_{H^1(\Gamma)}$. For all $J\geq 0$, define

$$f_J:=\sum_{j=0}^{J}\lambda_j a_j,$$

so that $f_J\to f$ in $H^1(\Gamma)$. For all $j_1<j_2$,

$$(I-P)^{-1/2}f_{j_2}-(I-P)^{-1/2}f_{j_1}=\sum_{j_1<j\leq j_2}\lambda_j(I-P)^{-1/2}a_j,$$

which entails, by Proposition 6.72,

$$
\begin{aligned}
\left\|(I-P)^{-1/2}f_{j_2}-(I-P)^{-1/2}f_{j_1}\right\|_{\dot{S}^{1,1}(\Gamma)} &\leq \sum_{j_1<j\leq j_2}|\lambda_j|\left\|(I-P)^{-1/2}a_j\right\|_{\dot{S}^{1,1}(\Gamma)}\\
&\leq C\sum_{j_1<j\leq j_2}|\lambda_j|.
\end{aligned}
$$

This shows that $((I-P)^{-1/2}f_j)_{j\geq 0}$ is a Cauchy sequence in $\dot{S}^{1,1}(\Gamma)$, and therefore converges to some function $g\in\dot{S}^{1,1}(\Gamma)$. Moreover, using Proposition 6.1 again,

$$\left\|g\right\|_{\dot{S}^{1,1}(\Gamma)}=\lim_{J\to+\infty}\left\|(I-P)^{-1/2}f_J\right\|_{\dot{S}^{1,1}(\Gamma)}\leq C\sum_{j=0}^{J}|\lambda_j|\leq 2C\left\|f\right\|_{H^1(\Gamma)}.$$

Furthermore, since $f_J\to f$ in $H^1(\Gamma)$, $d(I-P)^{-1/2}f_J\to d(I-P)^{-1/2}f$ in $L^1(E)$ (see [Rus01], Theorem 2.1). Since $d(I-P)^{-1/2}f_J\to dg$ in $L^1(E)$ by what we have just proved, $d(I-P)^{-1/2}f=dg$. As a consequence, $g=(I-P)^{-1/2}f\in\dot{S}^{1,1}(\Gamma)$ and

$$\left\|(I-P)^{-1/2}f\right\|_{\dot{S}^{1,1}(\Gamma)}\leq 2C\left\|f\right\|_{H^1(\Gamma)},$$

which concludes the proof of Theorem 2.15. □

6.2 Riesz transforms and Hardy spaces on edges

Apart from Theorem 2.15, it is also possible to establish that the Riesz transform maps $H^1(\Gamma)$ into a Hardy space on E, under assumptions (D) and (P_1), without assuming (2.21).
Indeed, since E, endowed with its distance d and its measure μ, is also a space of homogeneous type (see Section 2.1.1), we can define an atomic Hardy space on E. More precisely, an atom is a function $A \in L^2(E,\mu)$ (recall that A is antisymmetric), supported in a ball $B \subset E$ and satisfying

$$\sum_{(x,y)\in B} A(x,y)\mu_{xy} = 0 \text{ and } \|A\|_{L^2(E)} \leq \mu(B)^{-1/2}.$$

Define then $H^1(E)$ by the same procedure as for $H^1(\Gamma)$.

Our result is:

Theorem 6.4 *Assume that Γ satisfies (D) and (P_1). Then $d(I-P)^{-1/2}$ maps continuously $H^1(\Gamma)$ into $H^1(E)$.*

The proof goes through a duality argument. Let us introduce the $BMO(E)$ space. A function Φ on E belongs to $BMO(E)$ if, and only if, Φ is antisymmetric and

$$\|\Phi\|_{BMO(E)} := \left(\sup_{B\subset E} \frac{1}{\mu(B)} \sum_{(x,y)\in B} |\Phi(x,y) - \Phi_B|^2 \, d\mu_{xy}\right)^{1/2} < +\infty,$$

where the supremum is taken over all balls $B \subset E$ and, as usual,

$$\Phi_B := \frac{1}{\mu(B)} \sum_{(x,y)\in B} \Phi(x,y)\mu_{xy}.$$

Define also $CMO(E)$ as the closure in $BMO(E)$ of the space of antisymmetric functions on E with bounded support. Since E is a space of homogeneous type, one has ([CW77]):

Theorem 6.5
1. *The dual of $H^1(E)$ is $BMO(E)$.*
2. *The dual of $CMO(E)$ is $H^1(E)$.*

As in the proof of Theorem 2.15, Theorem 6.4 will be a consequence of:

Proposition 6.6 *Assume (D) and (P_1). Then there exists $C > 0$ such that, for all atom $a \in H^1(\Gamma)$,*

$$\left\|d(I-P)^{-1/2}a\right\|_{H^1(E)} \leq C.$$

Proof of Proposition 6.6: we argue similarly to the proof of [AT98], Chapter 4, Lemma 11 (see also Theorem 1 in [MR03]), and will therefore be very sketchy. Let a be an atom in $H^1(\Gamma)$ supported in a ball B. By assertion 2 in Theorem 6.5, it is enough to prove that, for all antisymmetric function Φ on E with bounded support,

$$\left|\sum_{(x,y)\in E} d(I-P)^{-1/2}a(x,y)\Phi(x,y)\mu_{xy}\right| \leq C\,\|\Phi\|_{BMO(E)}\,. \tag{6.89}$$

Since $d(I-P)^{-1/2}a \in L^1(E)$ and

$$\sum_{(x,y)\in E} d(I-P)^{-1/2}a(x,y)\mu_{xy} = 0,$$

one has

$$\sum_{(x,y)\in E} d(I-P)^{-1/2}a(x,y)\Phi(x,y)\mu_{xy} = \sum_{(x,y)\in E} d(I-P)^{-1/2}a(x,y)\left(\Phi(x,y)-\Phi_{2B}\right)\mu_{xy}, \quad (6.90)$$

and (6.89) is derived from (6.90) as in the proof of Lemma 11 in Chapter 4 of [AT98]. □

Here is another result about the boundedness of Riesz transforms on Hardy spaces. A function $u:\Gamma\to\mathbb{R}$ is said to be harmonic on Γ if and only if $(I-P)u(x)=0$ for all $x\in\Gamma$. Then:

Theorem 6.7 *Let $u:\Gamma\to\mathbb{R}$ be a harmonic function on Γ. Assume that there exist $x_0\in\Gamma$ and $C>0$ such that, for all $x\in\Gamma$,*

$$|u(x)| \le C(1+d(x_0,x)).$$

Define, for all functions f on Γ and all $x\in\Gamma$,

$$R_u(f)(x) = \sum_{y\in\Gamma} d(I-P)^{-1/2}f(x,y)du(x,y)\mu_{xy}.$$

Then R_u is $H^1(\Gamma)$ bounded.

Theorem 6.7 is a discrete counterpart of Theorem 1 in [MR03] and the proof goes through a duality argument, as in the proof of Theorem 1 in [MR03]. Indeed, the $H^1(\Gamma)-L^1(E)$ boundedness of $f\mapsto d(I-P)^{-1/2}f$ yields that R_u is $H^1(\Gamma)-L^1(\Gamma)$ bounded. Then, if $f\in H^1(\Gamma)$, one checks that

$$\sum_{x\in\Gamma} R_uf(x)m(x) = 0. \quad (6.91)$$

Indeed,

$$\begin{aligned}
\sum_{x\in\Gamma} R_uf(x)m(x) &= \sum_{x\in\Gamma} m(x)\sum_{y\sim x} d(I-P)^{-1/2}f(x,y)du(x,y)p(x,y)\\
&= \sum_{y\in\Gamma}\left(\sum_{x\in\Gamma} d(I-P)^{-1/2}f(x,y)du(x,y)p(x,y)m(x)\right)\\
&= \sum_{x,y} d(I-P)^{-1/2}f(x,y)du(x,y)\mu_{xy}\\
&= \langle d(I-P)^{-1/2}f, du\rangle_{L^2(E)}\\
&= \langle (I-P)^{-1/2}f, \delta du\rangle_{L^2(E)}\\
&= 0,
\end{aligned}$$

since $\delta du = 0$. Then, using (6.91), one proves, arguing as in [MR03], that, if a is an atom in $H^1(\Gamma)$, then, for all functions φ with bounded support on Γ,

$$\left|\sum_{x\in\Gamma} R_uf(x)\varphi(x)m(x)\right| \lesssim \|\varphi\|_{BMO(\Gamma)}.$$

The fact that $H^1(\Gamma)$ is the dual space of $CMO(\Gamma)$ then shows that

$$\|R_u a\|_{H^1(\Gamma)} \leq C,$$

and one concludes using the atomic decomposition for functions in $H^1(\Gamma)$. □

Let us make a few comments on Theorems 6.4 and 6.7. The conclusion of Theorem 6.4 says that, if $f \in H^1(\Gamma)$, then $d(I-P)^{-1/2}f$ has an atomic decomposition of the form

$$d(I-P)^{-1/2}f = \sum_{k\in\mathbb{N}} \lambda_k A_k$$

where $\sum_k |\lambda_k| \leq C \|f\|_{H^1(\Gamma)}$ and the A_k's are atoms in $H^1(E)$. However, one does not claim that each A_k is equal to da_k where a_k is an atom in $\dot{S}^{1,1}(\Gamma)$. In this sense, the conclusion of Theorem 6.4 is weaker than the one of Theorem 2.15. On the other hand, assumption (2.21) is not required in Theorem 6.4. Finally, Theorem 6.7 says that a scalar version of the Riesz transform is $H^1(\Gamma)$-bounded and does not require assumption (2.21) either.

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